Nonlinear Control

- Consider the following problem:

$\dot{x} = f(x,u), x \in R^n$ find $u = r(x)$ $\Rightarrow$ state feedback

$y = h(x), u \in R^n$ $u = \Gamma(y)$ $\Rightarrow$ output feedback

so that the closed loop system $\dot{x} = f(x, r(x))$ or $\dot{x} = f(x, \Gamma(x))$ exhibits $\underbrace{\text{desired stability}}_{\text{x is bounded \& goes to 0}}$ and $\underbrace{\text{performance}}_{\text{how it goes to setpoint}}$ characteristics.

- Why do we use nonlinear control :
  - Tracking, regulate state, state setpoint
  - Ensure the desired stability properties
  - Ensure the appropriate transients
  - Reduce the sensitivity to plant parameters

Nonlinear Control Vs. Linear Control

- Why not always use a linear controller ?
  - It just may not work.

Ex: $\dot{x} = x + u^3 \qquad x \in R$

When $u = 0$, the equilibrium point $x = 0$ is unstable.

Choose $u = -kx$.

Then $\dot{x} = x - k^3 x^3$.

We see that the system can't be made asymptotically stable at $x = 0$.

On the other hand, a nonlinear feedback does exist :

$$u(x) = -\sqrt[3]{kx}$$

Then $\dot{x} = x - kx = (1-k)x$

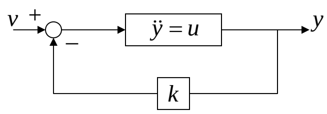
$\Downarrow$

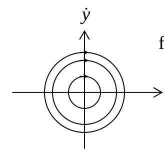
Asymptotically stable if $k > 1$.

Example

- Even if a linear feedback exists, nonlinear one may be better.

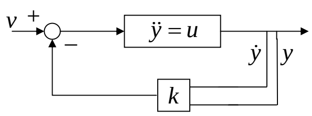
Ex: $\ddot{y} + ky = v$

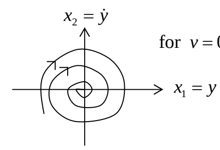
① 



for $v = 0$

$\Rightarrow \quad \ddot{y} + ky = v$

② 



for $v = 0$

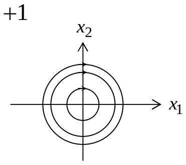
$\Rightarrow \quad \ddot{y} + k_1\dot{y} + k_2 y = v$

Example (continued)

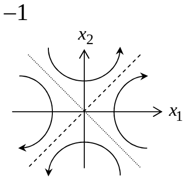
Let us use a nonlinear controller : To design it, consider again ①

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -kx_1$$

If $k = +1$

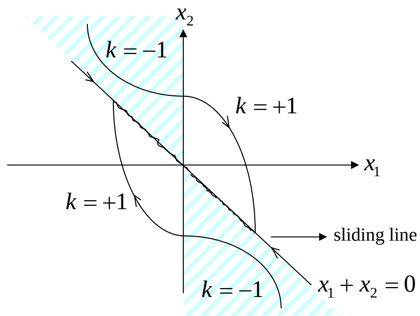


If $k = -1$



Example (continued)

Switch k from $+1$ to -1 appropriately and obtain a variable structure system.



$$k=\begin{cases}-1 & \text{if } x_1 s<0 \\ +1 & \text{if } x_1 s>0\end{cases}$$

where $s = x_1 + x_2$

Created a new trajectory: the system is insensitive to disturbance in the sliding regime ⇨ Variable structure control

The Tracking Problem

Consider the system:

$$\dot{x} = f(x) + u$$

Need to accomplish two control objectives:

1) Control Objective - make $x \to x_d$ (x_d is a desired trajectory), assuming $x_d, \dot{x}_d \in L_\infty$.
2) Hidden Control Objective - keep everything bounded (ie., $x, \dot{x}, u \in L_\infty$).

Need to make some assumptions first:

1) x is measureable.
2) if $x \in L_\infty$, then $f(x) \in L_\infty$.
3) x has a solution.
4) $x(0) \in L_\infty$.

The Tracking Problem (continued)

Let the tracking error, e, be defined as

$$e = x_d - x$$

$$\dot{e} = \dot{x}_d - \dot{x}$$

Now we can substitute for $\dot{x}$:

$$\dot{e} = \dot{x}_d - f(x) - u$$

Letting $\dot{e} = -ke$, we get:

$$u = \underbrace{\dot{x}_d - f(x)}_{\text{Feed Forward}} \underbrace{+ke}_{\text{Feedback}}$$

Now, solve the differential equation

$$e(t) = e(0)\exp(-kt)$$

Finally, insure all signals are bounded

$$\underbrace{x_d, \dot{x}_d \in L_\infty}_{\text{by assumption}}, e \in L_\infty \Rightarrow x \in L_\infty \Rightarrow f(x) \in L_\infty \Rightarrow u \in L_\infty \Rightarrow \dot{x} \in L_\infty$$

All signals are bounded!

Adaptive Control

Consider a linearly parameterizable function

$$f(x) = W(x)\Theta \quad \left(\text{for example } f(x) = \begin{bmatrix} x^2 & x^3\sin(x) \end{bmatrix} \cdot \begin{bmatrix} R \\ L \end{bmatrix} \right)$$

where $W(x)$ is known, and Θ is an unknown constant.

Let

$$\dot{e} = \dot{x}_d - W(x)\Theta - u \quad (1) \quad \Rightarrow \text{our control}$$

where

$$u = \dot{x}_d - W(x)\hat{\Theta} + ke \quad (2)$$

Let $\tilde{\Theta}$ be defined as

$$\tilde{\Theta} = \Theta - \hat{\Theta}$$

Now, combining (1) and (2), we get

$$\dot{e} = -ke - W(x)\tilde{\Theta}$$

Adaptive Control (continued)

Choose the Lyapunov candidate

$$V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{\Theta}^T\tilde{\Theta} = \frac{1}{2}z^T z, \text{ where } z = \begin{bmatrix} e \\ \tilde{\Theta} \end{bmatrix}$$

Q: Why is this a good candidate?
A: It is lower bounded (not necessarily by zero), radially unbounded in z, and positive definite in z.

Lemma: if

1) $V \geq 0$
2) $\dot{V} \leq -g(t)$, where $g(t) \geq 0$
3) $\dot{g}(t) \in L_\infty$

then $\lim_{t \to \infty} g(t) = 0$

Adaptive Control (continued)

With our candidate Lyapunov function

$$V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{\Theta}^T\tilde{\Theta}$$

Taking the derivative gives

$$\dot{V} = e\dot{e} + \tilde{\Theta}^T\dot{\tilde{\Theta}}$$

$$\dot{V} = e(-ke - W\tilde{\Theta}) - \tilde{\Theta}^T\dot{\hat{\Theta}}$$

$$\dot{V} = -ke^2 + \tilde{\Theta}^T(-W^T e - \dot{\hat{\Theta}})$$

Letting $\dot{\hat{\Theta}} = -W^T e$, we finally get

$$\dot{V} = -ke^2$$

Therefore $V \in L_\infty \Rightarrow \tilde{\Theta}, e, \hat{\Theta}, x \in L_\infty \Rightarrow u \in L_\infty \Rightarrow$ all signals are bounded!

For this problem

$$g(t) = ke^2 \text{ and } \dot{g}(t) = 2ke\dot{e} \in L_\infty$$

So our closed loop system is

$$\dot{e} = -ke - W\tilde{\Theta} \text{ and } \dot{\tilde{\Theta}} = W^T e$$

Q: So, does $\tilde{\Theta} \to 0$?
A: Not necessarily!

Robust (Sliding Mode) Control

Recall the system defined by the following:

$$\dot{x} = f(x) + u$$
$$e = x_d - x$$
$$\dot{e} = \dot{x}_d - f(x) - u$$

We can try to make several assumptions about the system:

1) $x_d, \dot{x}_d \in L_\infty$
2) if $x \in L_\infty$, then $f(x) \in L_\infty$.
3) $x \to x_d$ and all signals are bounded
4) $f(x)$ is linearly parameterizable (ie., $f(x) = W(x)\Theta$)

$\Rightarrow$ Adaptive control ONLY

5) $\underbrace{|f(x)|}_{\text{unknown dynamics}} < \underbrace{\rho(x)}_{\text{known bounding function}}$

$\Rightarrow$ We use this assumption for Robust (Sliding Mode) control ONLY!

Robust (Sliding Mode) Control (continued)

Now, let the control be

$$u = ke + \dot{x}_d + V_R$$

where V_R is a function that we can choose. Consider the three following functions

$$V_{R1} = \rho \frac{e}{|e|} \Rightarrow \text{Sliding mode}$$

$$V_{R2} = \frac{1}{\varepsilon} \rho^2 e \Rightarrow \text{Robust, high gain}$$

$$V_{R3} = \frac{\rho^2 e}{\rho|e| + \varepsilon} \Rightarrow \text{Robust, high frequency}$$

where $\varepsilon > 0$. We will consider each V_R separately.

Robust (Sliding Mode) Control (continued)

Let's try the first function

$$\dot{e} = -ke - f(x) - V_{R1}$$

Now, take a Lyapunov candidate

$$V = \tfrac{1}{2}e^2$$

$$\dot{V} = e\dot{e} = e(-ke - f(x) - V_{R1})$$

$$\dot{V} \le -ke^2 \underbrace{+|e||f(x)|}_{\text{from triangle inequality}} - eV_{R1}$$

$$\dot{V} \le -ke^2 + |e|\rho(x) - \frac{e^2}{|e|}\rho(x) \Rightarrow \text{used assumption 5 here!} \quad \left(|f(x)| < \rho(x)\right)$$

$$\dot{V} \le -ke^2 + |e|\rho(x) - |e|\rho(x) = -ke^2$$

$$e^2 = 2V$$

$$\dot{V} \le -2kV \Rightarrow \dot{V} + 2kV \le 0$$

$$\dot{V} + 2kV = -s(t), \text{ where } s(t) \ge 0$$

Robust (Sliding Mode) Control (continued)

Solving the differential equation, we get

$$V(t) = V(0)\exp(-2kt) - \exp(-2kt)\int_0^t \exp(2k\tau)s(\tau)d\tau$$

$$V(t) \le V(0)\exp(-2kt)$$

$$\tfrac{1}{2}e^2(t) \le \tfrac{1}{2}e^2(0)\exp(-2kt)$$

$$|e(t)| \le |e(0)|\exp(-kt)$$

So, the system is globally exponentially stable, and all signals are bounded!

Robust (Sliding Mode) Control (continued)

Now, let's try it with V_{R2} and the same Lyapunov function

$$\dot{e} = -ke - f(x) - V_{R2}$$

$$\dot{V} \le -ke^2 + |e|\rho(x) - eV_{R2}$$

$$\dot{V} \le -ke^2 + |e|\rho(x) - \frac{1}{\varepsilon}\rho^2(x)e^2$$

$$\dot{V} \le -ke^2 + |e|\rho(x)\left(1 - \frac{1}{\varepsilon}|e|\rho(x)\right)$$

$$\Rightarrow \text{if } |e|\rho(x) > \varepsilon, \text{ then } \dot{V} \le -ke^2$$

$$\Rightarrow \text{if } |e|\rho(x) \le \varepsilon, \text{ then } \dot{V} \le -ke^2 + \varepsilon$$

$$\dot{V} \le -2kV + \varepsilon$$

$$\dot{V} + 2kV = \varepsilon - s(t)$$

Robust (Sliding Mode) Control (continued)

Solving the differential equation yields

$$V(t) = V(0)\exp(-2kt) - \exp(-2kt)\int_0^t \exp(2k\tau)s(\tau)d\tau + \varepsilon\exp(-2kt)\underbrace{\int_0^t \exp(2k\tau)d\tau}_{\frac{1}{2k}(\exp(-2kt)-1)}$$

$$V(t) \le V(0)\exp(-2kt) + \frac{\varepsilon}{2k}\left[1 - \exp(-2kt)\right]$$

$$\tfrac{1}{2}e^2(t) \le \tfrac{1}{2}e^2(0)\exp(-2kt) + \frac{\varepsilon}{2k}\left[1 - \exp(-2kt)\right]$$

$$|e(t)| \le \sqrt{e^2(0)\exp(-2kt) + \frac{\varepsilon}{2k}\left[1 - \exp(-2kt)\right]}$$

The system is Globally Uniformly Ultimately Bounded (GUUB), and all signals are bounded.

Robust (Sliding Mode) Control (continued)

Using the third function, V_{R3}, we obtain similar results

$$\dot{V} \le -ke^2 + |e|\rho(x) - \frac{\rho^2 e^2}{\rho|e| + \varepsilon}$$

$$\dot{V} \le -ke^2 + \frac{|e|^2 \rho^2 + \varepsilon|e|\rho - \rho^2 e^2}{\rho|e| + \varepsilon}$$

$$\dot{V} \le -ke^2 + \varepsilon\left[\frac{\rho|e|}{\rho|e| + \varepsilon}\right]$$

$$\dot{V} \le -ke^2 + \varepsilon$$

As you can see, the solution to this equation will be the same as for V_{R2}

Learning Control

Let's take another look at the system from the previous control

$\dot{x} = f(x) + u,\ e = x_d - x,$ and $\dot{e} = \dot{x}_d - f(x) - u$

For each control type, we attempt to make different assumptions. Those assumptions eventually help us in the proof of stability and boundness of the system. For instance, we made the assumption that $f(x)$ was linearily parameterizable ($f(x) = W(x)\Theta$). For the Robust (Sliding Mode) control, we made the assumption that $f(x)$ was unknown, but that it could be bounded by some known function $\left(|f(x)| \le \rho(x)\right)$. For learning control, we make the assumption that $f(x)$ is periodic:

$f(x(t)) = f(x(t+T))$

Let $d(t) = f(x(t))$, that leaves us with

$\dot{e} = \dot{x}_d - d(t) - u,$ where d means "disturbance"

We also know, via our assumption that

$d(t) = d(t+T)$

Learning Control (continued)

Now, take the control to be

$$u = \dot{x}_d + ke - \hat{d}$$
$$\dot{e} = -ke - (d - \hat{d})$$

where

$$\tilde{d} = d - \hat{d}$$

Our task is to design $\hat{d}$. So, let's try

$$\hat{d}(t) = sat_\beta(\hat{d}(t-T)) - k_d e, \text{ where } sat_\beta(x) = \begin{Bmatrix} x \text{ for } |x| < \beta \\ \operatorname{sgn}(x)\beta \text{ for } |x| > \beta \end{Bmatrix}$$

We make the assumption that the magnitude of the disturbance, $d(t)$ is bounded:

$|d(t)| \le \beta$, where β is a constant

So, then we can say

$$d(t) = sat_\beta(d(t)) = sat_\beta(d(t-T))$$

$\tilde{d}(t) = sat_\beta(d(t-T)) - sat_\beta(\hat{d}(t-T)) + k_d e \Rightarrow$ use this in the stability proof!

Learning Control (continued)

We choose the following Lyapunov candidate to investigate stability:

$$V = \tfrac{1}{2}e^2 + \frac{1}{2k_d}\int_{t-T}^{t}\left[sat_\beta(d(\tau)) - sat_\beta(\hat{d}(\tau))\right]^2 d\tau$$

Leibniz's rule:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}$$

$V \ge 0 \Rightarrow$ Can you prove this?

$$\dot{V} = e(-ke - \tilde{d}(t)) + \frac{1}{2k_d}\left[sat_\beta(d(t)) - sat_\beta(\hat{d}(t))\right]^2$$
$$-\frac{1}{2k_d}\underbrace{\left[sat_\beta(d(t-T)) - sat_\beta(\hat{d}(t-T))\right]}_{\tilde{d} - k_d e}^2$$

$$\dot{V} = -ke^2 + \frac{1}{2k_d}\left[\left(sat_\beta(d(t)) - sat_\beta(\hat{d}(t))\right)^2 - \left(d(t) - \hat{d}(t)\right)^2\right] - \frac{k_d e^2}{2}$$

$$\dot{V} \le -ke^2$$

Math Note:

$$(x-y)^2 \ge \left(sat(x) - sat(y)\right)^2$$

So, $\dot{V} \le -g(t)$, for $g(t) \ge 0$ and

$$g(t) = ke^2 \Rightarrow \dot{g}(t) = 2ke\dot{e} \Rightarrow \dot{g}(t) \in L_\infty$$

So $g(t) \to 0$ and that means

$$V \in L_\infty \Rightarrow e \in L_\infty \Rightarrow x \in L_\infty \Rightarrow \hat{d}(t), \dot{e}, \dot{x}, u \in L_\infty \Rightarrow e \to 0$$

The Tracking Problem, Revisited

We want to build a tracking controller for the following system:

$$\dot{x} = -f(x) + u$$
$$e = x_d - x \Rightarrow x = x_d - e$$

where our controls are

$$\dot{e} = \dot{x}_d + f(x) - u$$
$$u = ke + V_{aux} + \dot{x}_d$$

which yields

$$\dot{e} = -ke + f(x) - V_{aux}$$

For this problem, we assume $f(x) \in C^1$ (where C^1 means differentiable)

$$\dot{e} = -ke + f(x_d) + \tilde{f} - V_{aux}, \text{ where } \tilde{f} = f(x) - f(x_d)$$

We also assume $\|\tilde{f}\| \le \|e\| \rho(\|x_d\|, \|e\|)$, where ρ is non-decreasing, and $\rho(x) \ge 0$. We also assume that $f(x)$ is known. So, we can say

$$V_{aux} = f(x_d)$$
$$\dot{e} = -ke + \tilde{f}$$

The Tracking Problem, Revisited (continued)

Let our Lyapunov candidate be

$$V = \tfrac{1}{2} e^2 \Rightarrow 2V = e^2$$
$$\dot{V} = e\dot{e} = -ke^2 + e\tilde{f}$$
$$\dot{V} \le -ke^2 + \|e\| \|\tilde{f}\|$$
$$\dot{V} \le -ke^2 + \|e\|^2 \rho(\|e\|)$$

Let $k = 1 + k_n$, then we have

$$\dot{V} \le -e^2 - (k_n - \rho(\|e\|))\|e\|^2$$
$$\dot{V} \le -e^2 \text{ if } k_n \ge \rho(\|e\|)$$
$$\dot{V} \le -2V \text{ if } k_n \ge \rho(\sqrt{2V})$$
$$V(t) \le V(0)\exp(-2t) \text{ if } k_n \ge \rho(\sqrt{2V(t)})$$
$$V(t) \le V(0)\exp(-2t) \text{ if } k_n \ge \rho(\sqrt{2V(0)})$$

So $k_n \ge \rho(\sqrt{2V(0)}) \ge \rho(\sqrt{2V(t)}) \Rightarrow$ true because ρ is non-decreasing

The Tracking Problem, Revisited (continued)

Now we can write

$$\tfrac{1}{2}e^2(t) \le \tfrac{1}{2}e^2(0)\exp(-2t) \text{ if } k_n \ge \rho\left(\sqrt{e^2(0)}\right)$$

$$\|e(t)\| \le \|e(0)\|\exp(-t) \text{ if } k_n \ge \rho(\|e(0)\|)$$

So, we have semi-global exponential tracking! It is semi-global (instead of just local) because we can, in theory, set k_n as high as we want. Also, as long as assumptions are met, all signals will remain bounded.

What if we assumed $f(x)$ was linearily parameterizable (ie., $f(x) = W(x)\Theta$)? Then we get

$$f(x_d) = W(x_d)\Theta$$

$$\dot{e} = -ke + W(x_d)\Theta + \tilde{f} - V_{aux}$$

Letting $V_{aux} = W(x_d)\hat{\Theta}$, would make $\dot{e} = -ke + W(\cdot)\tilde{\Theta} + \tilde{f}$, and we have

$$\dot{\hat{\Theta}} = W^T(\cdot)e$$

$$\dot{\tilde{\Theta}} = -W^T(\cdot)e$$

The Tracking Problem, Revisited (continued)

If we let our Lyapunov function be

$$V = \tfrac{1}{2}e^2 + \tfrac{1}{2}\tilde{\Theta}^T\tilde{\Theta}$$

we get

$$\dot{V} = -e^2 - \|e\|^2(k_n - \rho(\|e\|)) \underbrace{+ eW\tilde{\Theta} - \tilde{\Theta}^T\dot{\hat{\Theta}}}_{=0} \Rightarrow \text{recall that } \tilde{\Theta} = \Theta - \hat{\Theta} \text{ and } \dot{\tilde{\Theta}} = -\dot{\hat{\Theta}}$$

$$\dot{V} \le -e^2 \text{ if } k_n \ge \rho(\|e\|)$$

$$\dot{V} \le -e^2 \text{ if } k_n \ge \rho\left(\sqrt{2V(0)}\right) \Rightarrow \text{Be careful! We can't plug in } 2V \text{ for } e^2. \underbrace{\text{Why?}}_{V \text{ depends on } \Theta!}$$

Finally, we can show

$$\dot{V} \le -g(t), \text{ where } g(t) \ge 0$$

$$\dot{g}(t) = -2e\dot{e} \Rightarrow \dot{g}(t) \in L_\infty, \text{ so } \lim_{t\to\infty}|e(t)| = 0$$

We have semi-global asymptotic tracking.

Continuous Asymptotic Tracking

If we let

$$V = \tfrac{1}{2}x^2 + \tfrac{1}{2}\tilde{\Theta}^2 \Rightarrow \dot{V} = x\dot{x} - \tilde{\Theta}\dot{\hat{\Theta}} \Rightarrow \dot{V} = -x^2 + x\tilde{\Theta} - \tilde{\Theta}\dot{\hat{\Theta}}$$

having made $\dot{x} = -x + \tilde{\Theta}$ and $\tilde{\Theta} = \Theta - \hat{\Theta}$.

This let's us say

$$\dot{V} = -x^2 \text{ if } \dot{\hat{\Theta}} = x$$

Now try the new approach :

$$V = x^2 + P$$

$$\dot{V} = -x^2 + x\tilde{\Theta} + \dot{P}$$

where $\dot{P} = -x\tilde{\Theta}$, then $P = -\int_{t_0}^{t} x(\tau)\Theta(\tau)d\tau + \zeta_a$

Continuous Asymptotic Tracking (continued)

If you knew that $\dot{\hat{\Theta}} = -x$, then

$$P = -\int_{t_0}^{t} -\dot{\tilde{\Theta}}(\tau)\tilde{\Theta}(\tau)d\tau + \zeta_a = -\int_{t_0}^{t} \frac{d\left(\tfrac{1}{2}\tilde{\Theta}^2(\tau)\right)}{d\tau}d\tau + \zeta_a = \tfrac{1}{2}\tilde{\Theta}^2(t) - \tfrac{1}{2}\tilde{\Theta}^2(t_0) + \zeta_a$$

$V = -x^2$, so $\zeta_a = \tfrac{1}{2}\tilde{\Theta}^2(t_0)$

This solution is not unique even though we found it two different ways.

Continuous Asymptotic Tracking (continued)

Consider the system

$\dot{x} = \bar{f}(x) + \bar{g}(x)u \Rightarrow$ a scalar system where $\bar{f}$ and $\bar{g}$ are unknown

here, $\bar{g}(x) > 0$

We want $x(t) \to x_d(t)$, we can rewrite the system as

$m(x)\dot{x} + f(x) = u$, where $m(x) = \bar{g}(x)^{-1}$ and $f(x) = -\bar{g}(x)^{-1}\bar{f}(x)$

here, $m(x) > 0$

We make the following assumptions :

A1) $x_d \in C^3$

A2) $m(x), \dfrac{\partial m(x)}{\partial x}, \dfrac{\partial^2 m(x)}{\partial^2 x} \in L_\infty$ and $f(x), \dfrac{\partial f(x)}{\partial x}, \dfrac{\partial^2 f(x)}{\partial^2 x} \in L_\infty$

as long as $x \in L_\infty$

A3) $m(x_d), \dfrac{\partial m(x_d)}{\partial x_d}, \dfrac{\partial^2 m(x_d)}{\partial^2 x_d} \in L_\infty$ and $f(x_d), \dfrac{\partial f(x_d)}{\partial x_d}, \dfrac{\partial^2 f(x_d)}{\partial^2 x_d} \in L_\infty$

Continuous Asymptotic Tracking (continued)

Let our control be

$$u = (k_s + 1)e(t) - (k_s + 1)e(t_0) + \int_{t_0}^{t} \left((k_s + 1)\alpha e(\tau) + \beta \,\mathrm{sgn}\left(e(\tau)\right)\right) d\tau$$

where k_s, α, and β are positive constants. We see that $u(t_0) = 0$.

Here the error variable is defined as $e = x_d - x$

(Note : This controller is piece - wise continuous.)

Taking the derivative of u gives

$\dot{u}(t) = (k_s + 1)\dot{e}(t) + (k_s + 1)\alpha e(t) + \beta \,\mathrm{sgn}\left(e(t)\right)$

Let's define a new variable r as

$r = \dot{e} + \alpha e$

It can be shown that if $r \to 0$, then $e \to 0$, and if $r \in L_\infty$, then $e \in L_\infty$ (Why is this true?).

Continuous Asymptotic Tracking (continued)

Now, from our original system, we can write

$m(x)\dot{e} = m(x)\dot{x}_d + f(x) - u$ (where $\dot{e} = \dot{x}_d - \dot{x}$)

We also know

$\dot{r} = \ddot{e} + \alpha\dot{e}$

and we can then proceed as

$$m(x)\ddot{e} = -\dot{m}(x)\dot{e} + \dot{m}(x)\dot{x}_d + m(x)\ddot{x}_d + \dot{f}(x) - \dot{u}$$

$$m(x)\dot{r} = m(x)(\ddot{x}_d + \alpha\dot{e}) + \dot{m}(x)\dot{x} + \dot{f} - \dot{u}$$

$$m(x)\dot{r} = -\tfrac{1}{2}\dot{m}(x)r - e + N(x,\dot{x},t) - \dot{u}$$

where

$$N(x,\dot{x},t) = m(x)(\ddot{x}_d + \alpha\dot{e}) + \dot{m}(x)(\tfrac{1}{2}r + \dot{x}) + e + \dot{f}(x)$$

Substituting for $\dot{u}$ gives

$$m(x)\dot{r} = -\tfrac{1}{2}\dot{m}(x)r - e - (k_s + 1)r - \beta\,\mathrm{sgn}(e) + N(\cdot)$$

(Note that we can write $r = \dot{e} + \alpha e$)

Continuous Asymptotic Tracking (continued)

Let's study the stability of our control using the following Lyapunov candidate:

$$V = \tfrac{1}{2}e^2 + \tfrac{1}{2}m(x)r^2 + V_{new}$$

$$\dot{V} = e(-\alpha e + r) + r\left(-\tfrac{1}{2}\dot{m}(x)r - e - (k_s+1)r\right) + r\left(N(\cdot) - \beta\,\mathrm{sgn}(e)\right) + \tfrac{1}{2}\dot{m}(x)r^2 + \dot{V}_{new}$$

$$\dot{V} = -\alpha e^2 - r^2 + r\left(N(\cdot) - \beta\,\mathrm{sgn}(e) - k_s r\right) + \dot{V}_{new}$$

$$\dot{V} = -\alpha e^2 - r^2 + r\left(N_d(\cdot) - \beta\,\mathrm{sgn}(e)\right) + r(N(\cdot) - N_d(\cdot) - k_s r) + \dot{V}_{new}$$

Let us define a new variable L, as follows:

$$L(t) = r\left(N_d - \beta\,\mathrm{sgn}(e)\right) \qquad \{N_d = N(x,\dot{x},t) \mid x = x_d, \dot{x} = \dot{x}_d$$

We assume that $\tilde{N} = N - N_d$ can be bounded as follows

$$\left\|\tilde{N}(\cdot)\right\| \le \rho\left(\|z\|\right)\,\|z\|,\ z = \begin{bmatrix} e \\ r \end{bmatrix}$$

where, $\rho(\cdot)$, is a non-decreasing, positive, scalar function

So, due to the above assumptions $N_d, \dot{N}_d \in L_\infty$.

Continuous Asymptotic Tracking (continued)

Let $V_{new} = \zeta_b - \int_{t_0}^{t} L(\tau)d\tau$ (ζ_b is a positive constant) then, $\dot{V}_{new} = -L(t)$

where we still have to show that $V_{new} \geq 0$.

Substituting these definitions into the equaton for $\dot{V}$ we get

$$\dot{V} = -\alpha e^2 - r^2 + r(\tilde{N}(\cdot) - k_s r)$$

Now use the bound for $\tilde{N}$:

$$\dot{V} \leq -\alpha e^2 - r^2 \underbrace{+\|r\|\rho(\|z\|)\|z\| - k_s r^2}_{+\frac{\rho^2(\|z\|)}{4k_s}\|z\|^2 - \left(\sqrt{k_s}|r| + \frac{\rho(\|z\|)}{2\sqrt{k_s}}\|z\|\right)^2}$$

$$\dot{V} \leq -\lambda_3\|z\|^2 + \frac{\rho^2(\|z\|)\|z\|^2}{4k_s}$$

$$\dot{V} \leq -\left(\lambda_3 - \frac{\rho^2(\|z\|)}{4k_s}\right)\|z\|^2, \qquad \text{where } \lambda_3 = \min\{\alpha, 1\}$$

Continuous Asymptotic Tracking (continued)

We can also write V as: $V = \frac{1}{2}\begin{bmatrix} e & r \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & m(x) \end{bmatrix}\begin{bmatrix} e \\ r \end{bmatrix} + V_{new}$

where, $\lambda_1\|y\|^2 \leq V \leq \lambda_2(x)\|y\|^2$ and $y = \begin{bmatrix} z \\ \sqrt{V_{new}} \end{bmatrix}$

Now, let $\lambda_1 = \frac{1}{2}\min\{1, \underline{m}\}$ and $\lambda_2(x) = \max\left\{\frac{1}{2}\overline{m}(x), 1\right\}$

We then have

$$\dot{V} \leq -\kappa\|z\|^2 \text{ if } \lambda_3 \geq \frac{\rho^2(\|z\|)}{4k_s}, \text{ where } \kappa > 0$$

$$\dot{V} \leq -\kappa\|z\|^2 \text{ if } k_s \geq \frac{\rho^2(\|y\|)}{4\lambda_3}$$

Knowing that $\|y\| \leq \sqrt{\frac{V(t)}{\lambda_1}}$ we can write

$$\dot{V} \leq -\kappa\|z\|^2 \text{ if } k_s \geq \frac{\rho^2\left(\sqrt{\frac{V(t)}{\lambda_1}}\right)}{4\lambda_3} \Rightarrow \text{Here, we can replace } t \text{ with } t_0.$$

Continuous Asymptotic Tracking (continued)

So, we have Semi-Global Asymptotic tracking!

How do you know?

Remember our Lemma involving $\dot{V} \leq -g(t)$?

Recall our Lyapunov candidate

$$V = \tfrac{1}{2}e^2 + \tfrac{1}{2}m(x)r^2 + V_{new}$$

$$\dot{V} = \textit{negative terms} + L + \dot{V}_{new} \quad \text{and} \quad V_{new} \geq 0$$

$$L(t) = r(t)\left(N_d(t) - \beta \operatorname{sgn}\left(e(t)\right)\right)$$

$$\dot{V}_{new} = -L(t)$$

So, this gave us $\dot{V} = \textit{negative terms} + L(t) - L(t) = \textit{negative terms} \Rightarrow$ Asymptotic stability

Why not follow this procedure all the time?

Difficult to show that V_{new} is lower bounded by zero (i.e. the integral is always ≥ 0).

Continuous Asymptotic Tracking (continued)

So our result is only valid if $V_{new} \geq 0$.

$$V_{new} = \zeta_b - \int_{t_0}^{t}\left(\dot{e}(\tau) + \alpha e(\tau)\left(N_d(\tau) - \beta \operatorname{sgn}\left(e(\tau)\right)\right)\right)d\tau$$

$$\left.V_{new} = \zeta_b - \int_{t_0}^{t}\dot{e}(\tau)\left(N_d(\tau) - \beta \operatorname{sgn}(e(\tau))\right)d\tau - \int_{t_0}^{t}\alpha e(\tau)\left(N_d(\tau) - \beta \operatorname{sgn}(e(\tau))\right)d\tau\right\}\text{Expanded}$$

Remember $L = r\left(N_d - \beta \operatorname{sgn}\left(e\right)\right)$. We now show that if β is selected as

$$\beta > \left|N_d(t)\right| + \frac{1}{\alpha}\left|\dot{N}_d(t)\right|,$$

then $\int_{t_0}^{t} L(\tau)d\tau \leq \zeta_b$.

So $\zeta_b = \beta\left|e(t_0)\right| - e(t_0)N_d(t_0) \geq 0$

Continuous Asymptotic Tracking (continued)

Working with just the integral:

$$\int_{t_0}^{t} L(\tau)d\tau = \underbrace{\int_{t_0}^{t} \frac{de(\tau)}{d\tau} N_d(\tau)d\tau}_{\text{Integrate by parts}} - \beta\int_{t_0}^{t} \frac{de(\tau)}{d\tau}\text{sgn}\left(e(\tau)\right)d\tau + \int_{t_0}^{t} \alpha e(\tau)\left(N_d(\tau) - \beta\,\text{sgn}\left(e(\tau)\right)\right)d\tau$$

$$\int_{t_0}^{t} L(\tau)d\tau = e(\tau)N_d(\tau)\Big|_{t_0}^{t} - \int_{t_0}^{t} e(\tau)\frac{dN_d(\tau)}{d\tau}d\tau - \beta\left|e(\tau)\right|\Big|_{t_0}^{t} + \int_{t_0}^{t} \alpha e(\tau)\left(N_d(\tau) - \beta\,\text{sgn}\left(e(\tau)\right)\right)d\tau$$

Note: $\dfrac{d\sqrt{x^2}}{dt} = \dfrac{\frac{1}{2}\left(2x\dot{x}\right)}{\sqrt{x^2}} = \dfrac{x}{|x|}\dot{x} = \text{sgn}(x)\dot{x}$

$$\int_{t_0}^{t} L(\tau)d\tau = \left[\overbrace{e(t)N_d(t) - \beta\left|e(t)\right|}^{\alpha_1} - \overbrace{e(t_0)N_d(t_0) + \beta\left|e(t_0)\right|}^{\zeta_b}\right]$$

α_1 and ζ_b are positive since $\beta > \left|N_d(t)\right|$

$$+\int_{t_0}^{t} \alpha e(\tau)\left[N_d(\tau) - \frac{1}{\alpha}\frac{dN_d(\tau)}{d\tau} - \beta\,\text{sgn}\left(e(\tau)\right)\right]d\tau$$

This term is always negative

So, we have $\displaystyle\int_{t_0}^{t} L(\tau)d\tau \le \zeta_b$

Feedback Linearization Problem

Consider the system

$M(q)\ddot{q} + V_m(q,\dot{q})\dot{q} + G(q) + F(\dot{q}) = \tau$, where $M(q)$ is positive definite, symmetric.

$x^T\left(\tfrac{1}{2}\dot{M}(q) - V_m(q,\dot{q})\right)x = 0$

$e = q_d - q$

We could rewrite the system as

$M(q)\ddot{q} + V_m(q,\dot{q})\dot{q} + N(q,\dot{q}) = \tau \qquad (N(q,\dot{q}) = G(q) + F(\dot{q}))$

$M\ddot{e} = M\ddot{q}_d + V_m\dot{q} + N - \tau$

If we know everything about the system (the model), we can write

$\tau = M(\ddot{q}_d + k_v\dot{e} + k_p e) + V_m\dot{q} + N$

$\ddot{e} + k_v\dot{e} + k_p e = 0$

What if we try

$\tau = \hat{M}(\ddot{q}_d + k_v\dot{e} + k_p e) + \hat{V}_m\dot{q} + \hat{N}$

$M\ddot{e} = -\hat{M}(k_v\dot{e} + k_p e) + (V_m - \hat{V}_m)\dot{q} + N - \hat{N} + (M - \hat{M})\ddot{q}_d$

Feedback Linearization Problem (continued)

Continuing from previous slide:

$$\ddot{e} = -k_v\dot{e} - k_p e + (I - M^{-1}\hat{M})(k_v\dot{e} + k_p e) + M^{-1}(\tilde{M}\ddot{q}_d + \tilde{V}_m\dot{q} + \tilde{N})$$

where $\tilde{M} = M - \hat{M}$, $\tilde{V}_m = V_m - \hat{V}_m$ and $\tilde{N} = N - \hat{N}$

$$\ddot{e} + k_v\dot{e} + k_p e = f(M, \tilde{M}, \tilde{V}_m, \tilde{N}) = f(e, \dot{e}, q_d, \dot{q}_d, \ddot{q}_d, q, \dot{q}) \Rightarrow \text{Not good. Why?}$$

Let's try something else. Define

$$r = \dot{e} + \alpha e$$

$$\dot{r} = \ddot{e} + \alpha\dot{e}$$

Multiplying through by M gives

$$M\dot{r} = M\ddot{e} + M\alpha\dot{e}$$

$$M\dot{r} = M(\ddot{q}_d + \alpha\dot{e}) + V_m\dot{q} + N - \tau$$

$$M\dot{r} = -V_m r + \left(M(\ddot{q}_d + \alpha\dot{e}) + V_m(\dot{q}_d + \alpha e) + N\right) - \tau$$

$$M\dot{r} = -V_m r + Y(\dot{q}, q, \ddot{q}_d, \dot{q}_d, q_d)\Theta - \tau$$

Design your control, letting $\tau = Y\hat{\Theta} + kr$ and $\dot{\hat{\Theta}} = Y^T r$. Now, we can write

$$M\dot{r} = -V_m r - kr + Y\tilde{\Theta}, \text{ where } \dot{\tilde{\Theta}} = -Y^T r,\ \tilde{\Theta} = \Theta - \hat{\Theta}, \text{ and } \dot{\tilde{\Theta}} = -\dot{\hat{\Theta}}.$$

Feedback Linearization Problem (continued)

Our Lyapunov candidate can be selected to be

$$V = \tfrac{1}{2} r^T M r + \tfrac{1}{2}\tilde{\Theta}^T\tilde{\Theta}$$

which gives

$$\dot{V} = r^T M\dot{r} + \tfrac{1}{2} r^T \dot{M} r - \tilde{\Theta}^T\dot{\hat{\Theta}}$$

$$\dot{V} = \tfrac{1}{2}\dot{r}^T M r + \tfrac{1}{2} r^T M\dot{r} + \tfrac{1}{2} r^T \dot{M} r + \tfrac{1}{2}\dot{\tilde{\Theta}}^T\tilde{\Theta} + \tfrac{1}{2}\tilde{\Theta}^T\dot{\tilde{\Theta}} - \tilde{\Theta}^T\dot{\hat{\Theta}}$$

$$\dot{V} = r^T(-V_m r - kr + Y\tilde{\Theta}) - \tilde{\Theta}^T Y^T r + \tfrac{1}{2} r^T \dot{M} r$$

$$\dot{V} = -r^T kr = -g(t) \Rightarrow \text{recall that } M\|r\|^2 \le r^T M r$$

So, all signals are bounded, and $r \to 0$ (due to our stability lemma). Notice that this way did not feedback linearize the system like the previous one.

Previous Problem Using a Robust Approach

For the previous system, we want to apply a robust control:

$$M\dot{r} = -V_m r + W - \tau$$

$$W = M(\ddot{q}_d + \alpha\dot{e}) + V_m(\dot{q}_d + \alpha\dot{e}) + N(q,\dot{q})$$

We made the assumption that $M(q)$ was p.d. symmetric and $x^T(\frac{1}{2}\dot{M} - V_m)x = 0$.

Let our control be

$$\tau = kr + V_R, \text{ where we choose from } V_{R1} = \frac{\rho^2 r}{\varepsilon},\ V_{R2} = \frac{\rho^2 r}{\|r\|\rho + \varepsilon},\text{ or } V_{R3} = \frac{\rho r}{\|r\|}$$

So, our system can be written

$$M\dot{r} = -kr - V_m r + W - V_R$$

Where $\rho > \|W\|$

Choose the Lyapunov candidate to be

$$V = \frac{1}{2} r^T M r$$

Taking the derivative gives

$$\dot{V} = -r^T kr + r^T(W - V_R)$$

Previous Problem Using a Robust Approach

Continuing from the previous slide:

$$\dot{V} \le -\lambda_{\min}\{k\}\|r\|^2 + \underbrace{\|r\|\rho - \frac{\rho^2\|r\|^2}{\varepsilon}}_{\|r\|\rho\left(1-\frac{\rho\|r\|}{\varepsilon}\right)\le\varepsilon}$$

$$\dot{V} \le -\lambda_{\min}\{k\}\|r\|^2 + \varepsilon$$

Since M is p.d. symmetric, we can write

$$\frac{1}{2}m_1\|r\|^2 \le V \le \frac{1}{2}m_2\|r\|^2 \qquad (m_1, m_2 \text{ are constants})$$

Where the assumption $m_1\|x\|^2 \le x^T M(q)x \le m_2\|x\|^2$ was used.

Let $\gamma = \frac{2\lambda_{\min}\{k\}}{m_2}$, which leads to

$$\dot{V} \le \frac{-2\lambda_{\min}\{k\}}{m_2}V + \varepsilon \Rightarrow \dot{V} \le -\gamma V + \varepsilon$$

$$V(t) \le V(0)\exp(-\gamma t) + \left(1 - \exp(-\gamma t)\right)\frac{\varepsilon}{\gamma}$$

Therefore, the system is GUUB.

On a practical note, high gains cause noise to corrupt actual experiments.

Observers

Consider the linear system

Plant

$\dot{x} = Ax + Bu$

$y = Cx$

Observer

$\dot{\hat{x}} = A\hat{x} + Bu + L\tilde{y}$

$\hat{y} = C\hat{x}$

where $\tilde{y} = y - \hat{y}$

$u_1 = -k\hat{x} \Rightarrow$ for the plant + the observer

$u_2 = -kx \Rightarrow$ for just the plant

Note: For the observer to work in the system above, you will need exact model knowledge!

The separation principle (linear systems ONLY) says that u_2 for the plant works just like u_1 for the plant and the observer.

Observers (continued)

What about a nonlinear system? Consider the system

$\dot{x} = f(x) + g(x)u \;\Rightarrow$ nonlinear, so we can't use the Separation Principle!

$y = h(x) \;\Rightarrow$ suppose you can only measure this - you'll want an observer!

$$\left.\begin{aligned} \dot{\hat{x}} &= \Pi_1(\hat{x}, u, y) \\ u &= \Pi_2(\hat{x}) \end{aligned}\right\} \Pi_1(\cdot) \text{ and } \Pi_2(\cdot) \text{ are designed}$$

Then, you could try

$$\left.\begin{aligned} \dot{\hat{x}} &= f(\hat{x}) + g(\hat{x})u + L\tilde{y} \\ \hat{y} &= h(\hat{x}) \\ u &= -k\hat{x} \end{aligned}\right\} \text{Difficult to prove stability result}$$

Observers (continued)

Let's try to develop an observer for the nonlinear system of the form

$\ddot{x} = f(x,\dot{x}) + u$

Our control objective then is to force $x \to x_d$ when only x is measureable.

The nonlinear system above can be represented by two cases:

Case 1) $f(\cdot)$ is known, but unmeasurable e.g. $f(x,\dot{x}) = x^2\dot{x}^4$

Case 2) $f(\cdot)$ is uncertain and unmeasurable, e.g. $f(x,\dot{x}) = ax^2\dot{x}^4$ { a is unknown.

For Case 1, we can estimate $\dot{x}$:

$\ddot{\hat{x}} = f(x,\dot{\hat{x}}) + u \quad \Rightarrow \quad$ open loop observer

$\ddot{\tilde{x}} = f(x,\dot{x}) - f(x,\dot{\hat{x}})$, where $\tilde{x} = x - \hat{x}$

Recall: $\ddot{\tilde{x}} = \ddot{x} - \ddot{\hat{x}}$

What about letting

$\underbrace{\dot{\hat{x}} = p + k_{01}\tilde{x} \text{ and } \dot{p} = f(x,\dot{\hat{x}}) + u + k_{02}\tilde{x}}_{\text{closed loop observer}}$

Observers (continued)

Now we get

$\ddot{\hat{x}} = \dot{p} + k_{01}\dot{\tilde{x}} = f(x,\dot{\hat{x}}) + u + k_{01}\dot{\tilde{x}} + k_{02}\tilde{x}$ } After differentiating $\dot{\hat{x}}$

$\ddot{\tilde{x}} + k_{01}\dot{\tilde{x}} + k_{02}\tilde{x} = f(x,\dot{x}) - f(x,\dot{\hat{x}}) = \tilde{f}$ } From $\ddot{\tilde{x}} = \ddot{x} - \ddot{\hat{x}}$

Suppose we can show

$\left\|\tilde{f}\right\| \le \zeta_1\left\|\tilde{x}\right\| + \zeta_2\left\|\dot{\tilde{x}}\right\| \quad \Rightarrow \quad$ linear bound

Then

$\ddot{\tilde{x}} = -k_{01}\dot{\tilde{x}} - k_{02}\tilde{x} + \tilde{f} \quad \Rightarrow \quad$ so, $V = function(\tilde{x},\dot{\tilde{x}})$ } This makes the problem complicated

Instead, let's try a change of variables. Define the following:

$s = \dot{\tilde{x}} + \alpha\tilde{x}$ and $\dot{s} = \ddot{\tilde{x}} + \alpha\dot{\tilde{x}}$

Which gives

$\dot{s} = -k_{01}\dot{\tilde{x}} - k_{02}\tilde{x} + \tilde{f} + \alpha\dot{\tilde{x}}$

Make $k_{01} = \alpha + \bar{k}$ and $k_{02} = \alpha\bar{k} + 1$, then

$\dot{s} = -\bar{k}s + \tilde{f} - \tilde{x}$

$\dot{\tilde{x}} = -\alpha\tilde{x} + s \quad \Rightarrow \quad$ we just rearranged!

} Two first order systems instead of one second order system

Observers (continued)

Consider the Lyapunov candidate:

$$V = \tfrac{1}{2}\tilde{x}^2 + \tfrac{1}{2}s^2 = \tfrac{1}{2}z^T z \qquad \text{(Here } z = [\tilde{x}, s]^T\text{)}$$

where

$$\dot{V} = \tilde{x}(-\alpha\tilde{x} + s) + s(-\bar{k}s - \tilde{x} + \tilde{f})$$

$$\dot{V} = -\alpha\tilde{x}^2 - \bar{k}s^2 + s\tilde{f}$$

say $\|\tilde{f}\| \le \bar{\zeta}_1\|\tilde{x}\| + \bar{\zeta}_2\|s\|$, then

$$\dot{V} \le -\alpha\tilde{x}^2 - \bar{k}s^2 + \bar{\zeta}_1\|\tilde{x}\|\|s\| + \bar{\zeta}_2\|s\|^2 \quad \Rightarrow \text{ Note: We used } \dot{\tilde{x}} = s - \alpha\tilde{x}$$

We can use the property

$$|x||y| \le \frac{|x|^2}{\delta} + \delta|y|^2 \quad \Big\} \text{ Where } \delta \text{ is some positive constant}$$

which allows us to write

$$\dot{V} \le -(\alpha - \bar{\zeta}_1\delta)\|\tilde{x}\|^2 - \left(\bar{k} - \bar{\zeta}_2 - \frac{\bar{\zeta}_1}{\delta}\right)\|s\|^2 \quad \Rightarrow \Big\{ \text{If } \alpha \text{ and } \bar{k} \text{ are selected large enough, negative definite, so } x \text{ and } s \to 0!$$

All signals bounded! (Can you show this?) Here we assume that $x, \dot{x} \in L_\infty$

Observers (continued)

Suppose we redefine $\hat{f}$:

$$\hat{f}(\cdot) = \hat{f}(\hat{x}, \dot{\hat{x}}) \quad \Rightarrow \text{so now it depends on } \hat{x} \text{ instead of } x$$

If $f \in c^1$, then we can use the Mean Value Theorem to state

$$\left\|\tilde{f}(x, \hat{x}, \dot{x}, \dot{\hat{x}})\right\| \le \rho(x, \dot{x}, \tilde{x}, \dot{\hat{x}})\left\|\begin{matrix}\tilde{x}\\ \dot{\tilde{x}}\end{matrix}\right\|$$

where $\tilde{f} = f(x, \dot{x}) - \hat{f}(\hat{x}, \dot{\hat{x}})$.

We can then write

$$\|\tilde{f}\| \le \bar{\rho}(x, \dot{x}, \tilde{x}, s)\left\|\begin{matrix}\tilde{x}\\ s\end{matrix}\right\|$$

$$\|\tilde{f}\| \le \bar{\rho}_1(\tilde{x}, s)\left\|\begin{matrix}\tilde{x}\\ s\end{matrix}\right\|$$

Observers (continued)

For the observer problem:

$$\dot{V} \le -\alpha\tilde{x}^2 - \bar{k}s^2 + \|s\|\bar{\rho}_1(\tilde{x},s)\left\|\begin{matrix}\tilde{x}\\ s\end{matrix}\right\| \quad \Rightarrow \quad \text{remember we found that } \dot{V} = -\alpha\tilde{x}^2 - \bar{k}s^2 + s\tilde{f}$$

let $\alpha = 1$ and $\bar{k} = k_n + 1$. Then,

$$\dot{V} = -\tilde{x}^2 - s^2 + \underbrace{\|z\|\bar{\rho}_1(\|z\|)\|s\| - k_n\|s\|^2}_{\le \frac{\bar{\rho}_1^2(\|z\|)\|z\|^2}{k_n}}, \; z = \left\|\begin{matrix}\tilde{x}\\ s\end{matrix}\right\|$$

Define

$$N = \|z\|\bar{\rho}_1(\cdot) \text{ and } L = N\|s\| - k_n\|s\|^2$$

There are two different cases:

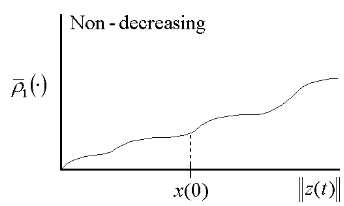
Case 1) $k_n\|s\|^2 > N\|s\| \quad \Rightarrow \quad L < 0$

Case 2) $k_n\|s\|^2 < N\|s\|$ so, $\|s\| < \frac{N}{k_n} \quad \Rightarrow \quad L \le \frac{N^2}{k_n} - k_n\|s\|^2$

then $L < \frac{N^2}{k_n}$

Observers (continued)

So,



$$\dot{V} \le -\|z\|^2 + \frac{\bar{\rho}_1^2(\cdot)\|z\|^2}{k_n}$$

$$\dot{V} \le -\left(1 - \frac{\bar{\rho}_1^2(\cdot)}{k_n}\right)\|z\|^2$$

$\dot{V} \le -\beta\|z\|^2$ if $k_n > \bar{\rho}_1^2(\|z\|)$, where β is a positive constant.

Recall that $V = \frac{1}{2}\|z\|^2$. So, we can write

$$\dot{V} \le -2\beta V \text{ if } k_n > \bar{\rho}_1^2\left(\sqrt{2V}\right)$$

$$V(t) \le V(0)\exp(-2\beta V) \text{ if } k_n > \bar{\rho}_1^2\left(\sqrt{2V(t)}\right)$$

$$\frac{1}{2}\|z(t)\|^2 \le \frac{1}{2}\|z(0)\|^2\exp(-2\beta t) \text{ for } k_n > \bar{\rho}_1^2\left(\sqrt{2V(0)}\right) > \bar{\rho}_1^2\left(\sqrt{2V(t)}\right)$$

$$\|z(t)\| \le \|z(0)\|\exp(-\beta t) \text{ for } k_n > \bar{\rho}_1(\|z(0)\|)$$

This gives us a semi-global exponential result! Why not global?

Combining Observers & Controllers

Can we develop a combined observer/controller for the previous system?

$\ddot{x} = f(x,\dot{x}) + u$

Well, we assume $f(x,\dot{x}) \in L_\infty$ if $x,\dot{x} \in L_\infty \quad \Rightarrow$ but we can't measure $\dot{x}$!

The observer/controller is more complex since all signals must be shown to be bounded.

We can choose from two different error systems

Case 1) $e = x_d - \hat{x} \Rightarrow \dot{e} = \dot{x}_d - \dot{\hat{x}}$

Case 2) $e_1 = x_d - x \Rightarrow \dot{e}_1 = \dot{x}_d - \dot{x}$

Let's use Case 1 since $\dot{e}$ is measureable. Rename it e_1 (now $e = e_1$).

This gives us

$\dot{e}_1 = \dot{x}_d - \dot{\hat{x}} = \dot{x}_d - p - k_{01}\tilde{x} = \dot{x}_d - k_{01}\tilde{x} - p_d + \tilde{p},$ where $\tilde{p} = p_d - p$

Letting $p_d = \dot{x}_d - k_{01}\tilde{x} + k_{c1}e_1$ gives

$\dot{e}_1 = -k_{c1}e_1 + \tilde{p}$

Combining Observers & Controllers (continued)

Recognizing that $\dot{\tilde{p}} = \dot{p}_d - \dot{p},$ we can write

$\dot{\tilde{p}} = \ddot{x}_d - k_{01}\dot{\tilde{x}} + k_{c1}\dot{e}_1 - \left(\hat{f}(\cdot) + k_{02}\tilde{x} + u\right)$

$\dot{\tilde{p}} = \underbrace{\left[\ddot{x}_d + k_{c1}\left(k_{c1}\dot{e}_1 + \tilde{p}\right) - \hat{f}(\cdot) - k_{02}\tilde{x}\right]}_{=W_1 \rightarrow \text{measureable}} - k_{01}\dot{\tilde{x}} - u$

We can design the control as follows

$$\left.\begin{aligned} u &= W_1 + k_{c2}\tilde{p} + e_1 + V_{aux} \\ \dot{\tilde{p}} &= -k_{c2}\tilde{p} - e_1 - k_{01}\dot{\tilde{x}} - V_{aux} \end{aligned}\right\} V_{aux} \text{ is designed during the stability proof}$$

Taking the following Lyapunov functions

$V_c = \frac{1}{2}e_1^2 + \frac{1}{2}\tilde{p}^2$

$\dot{V}_c = -k_{c1}e_1^2 + e_1\tilde{p} - k_{c2}\tilde{p}^2 - e_1\tilde{p} + \tilde{p}(-k_{01}\dot{\tilde{x}} - V_{aux})$

$V_o = \frac{1}{2}\tilde{x}^2 + \frac{1}{2}s^2$

$\dot{V}_o = -\alpha\tilde{x}^2 - \bar{k}s^2 + s\tilde{f}$

where the combined Lyapunov function can be written

$V = V_o + V_c$

Combining Observers & Controllers (continued)

Now, we can write the derivative of the combined Lyapunov function as

$$\dot{V} = \underbrace{-\alpha\tilde{x}^2 - \bar{k}s^2 - k_{c1}e_1^{\,2} - k_{c2}\,\tilde{p}^2}_{\text{good terms (Why?)}} \overbrace{\underbrace{+s\tilde{f} - \tilde{p}k_{01}\dot{\hat{x}}}_{\text{bad terms (Why?)}} \underbrace{-\tilde{p}V_{aux}}_{\text{injected term}}}^{L}$$

$$\tilde{p}k_{01}\dot{\hat{x}} = \tilde{p}k_{01}(-\alpha\tilde{x} + s) = \underbrace{-\alpha k_{01}\tilde{p}\tilde{x}}_{\text{measureable}} \; \underbrace{+\tilde{p}k_{01}s}_{\text{unmeasureable}}$$

Letting $V_{aux} = V_{aux1} + \alpha k_{01}\tilde{x} + k_{n1}k_{01}^2\tilde{p}$ lets us write $\}$ V_{aux1} will be designed later

$$L = s\tilde{f} - \tilde{p}k_{01}(-\alpha\tilde{x} + s) - \tilde{p}V_{aux1} - \tilde{p}(\alpha k_{01}\tilde{x} + k_{n1}k_{01}^2\tilde{p})$$

Also, we can say

$$\left.\begin{aligned} L &\le (\cdots) + k_{01}\left|\tilde{p}\right|\left|s\right| - k_{n1}k_{01}^2\left|\tilde{p}\right|^2 \\ L &\le (\cdots) + \frac{\left|s\right|^2}{k_{n1}} \end{aligned}\right\} \text{Nonlinear damping on one term}$$

So, we can write

$$\dot{V} \le -\alpha\tilde{x}^2 - \left(\bar{k} - \frac{1}{k_{n1}}\right)s^2 - k_{c1}e_1^{\,2} - k_{c2}\,\tilde{p}^2 + s\tilde{f} - \tilde{p}V_{aux1}$$

Combining Observers & Controllers (continued)

Recall that

$$\tilde{f} = f(x,\dot{x}) - \hat{f}(\hat{x},\dot{\hat{x}})$$

Let's assume that $f \in c^1$, then

$$\left\|\tilde{f}\right\| \le \rho(x,\dot{x},\hat{x},\dot{\hat{x}})\left\|z\right\|, \text{ where } \left\|z\right\| = \left\|\begin{matrix}\tilde{x}\\ s\end{matrix}\right\|$$

It can be shown that

$$\rho(x,\dot{x},\hat{x},\dot{\hat{x}}) \le \rho_1(\hat{x},\dot{\hat{x}},\tilde{x},\dot{\tilde{x}}) \le \rho_2(\hat{x},\dot{\hat{x}},\tilde{x},s) \le \rho_3(x_d,e_1,\dot{\hat{x}},\tilde{x},s) \le \rho_4(x_d,\dot{x}_d,e_1,\dot{e}_1,\tilde{x},s)...$$
$$... \le \rho_5(x_d,\dot{x}_d,e_1,\tilde{p},\tilde{x},s) \le \rho_6(e_1,\tilde{p},\tilde{x},s)$$

Then we can write

$$\dot{V} \le -\alpha\tilde{x}^2 - \left(\bar{k} - \frac{1}{k_{n1}}\right)s^2 - k_{c1}e_1^{\,2} - k_{c2}\,\tilde{p}^2 + \left\|s\right\|\rho_6(\left\|e_1\right\|,\left\|\tilde{p}\right\|,\left\|\tilde{x}\right\|,\left\|s\right\|)\left\|z\right\| \;\Rightarrow\; \text{we let } V_{aux1} = 0$$

Combining Observers & Controllers (continued)

From the previous slide:

$$\dot{V} \le -\alpha\tilde{x}^2 - \left(\bar{k} - \frac{1}{k_{n1}}\right)s^2 - k_{c1}e_1^{\ 2} - k_{c2}\tilde{p}^2 + \|s\|\rho_6(\|\bar{z}\|)\|\bar{z}\|,$$

where $\|\bar{z}\| = \left\|\begin{matrix} z \\ e_1 \\ \tilde{p} \end{matrix}\right\|$

If we let $\alpha, k_{c1}, k_{c2} = 1$ and $\bar{k} - \dfrac{1}{k_{n1}} = k_F + 1,$ we can write

$$\dot{V} \le -\|\bar{z}\|^2 + \underbrace{\|s\|\rho_6(\|\bar{z}\|)\|\bar{z}\| - k_F\|s\|^2}_{N_3}$$

If $k_F\|s\| > \rho_6(\|\bar{z}\|)\|\bar{z}\|,$ then $\quad N_3 = \|s\|\left(\rho_6(\|\bar{z}\|)\|\bar{z}\| - k_F\|s\|\right) \le 0$

If $k_F\|s\| \le \rho_6(\|\bar{z}\|)\|\bar{z}\|,$ then $\quad N_3 \le \dfrac{\rho_6^{\ 2}\left(\|\bar{z}\|\right)\|\bar{z}\|^2}{k_F}$

which gives, $\quad \dot{V} \le -\|\bar{z}\|^2 + \dfrac{\rho_6^{\ 2}(\|\bar{z}\|)\|\bar{z}\|^2}{k_F}$

Combining Observers & Controllers (continued)

From previous slide:

$$\dot{V} \le -\|\bar{z}\|^2 + \frac{\rho_6^{\ 2}(\|\bar{z}\|)\|\bar{z}\|^2}{k_F}$$

So, $\quad \dot{V} \le -\left(1 - \dfrac{\rho_6^{\ 2}(\|\bar{z}\|)}{k_F}\right)\|\bar{z}\|^2$

$$\dot{V} \le -\beta\|\bar{z}\|^2 \ \ if\ \ k_F \ge \rho_6^{\ 2}\left(\|\bar{z}\|\right)$$

Remembering that $V = \frac{1}{2}\|\bar{z}\|^2$, we can say

$$\dot{V} \le -2\beta V \text{ if } k_F \ge \rho_6^{\ 2}\left(\sqrt{2V}\right)$$

$$V(t) \le V(0)\exp(-2\beta t) \text{ if } k_F \ge \rho_6^{\ 2}\left(\sqrt{2V(0)}\right)$$

Now, we can write

$$\tfrac{1}{2}\|\bar{z}(t)\|^2 \le \tfrac{1}{2}\|\bar{z}(0)\|^2\exp(-2\beta t) \text{ if } k_F > \rho_6^{\ 2}\left(\|\bar{z}(0)\|\right)$$

$$\|\bar{z}(t)\| \le \|\bar{z}(0)\|\exp(-\beta t) \ \text{ if } \ k_F > \rho_6^{\ 2}\left(\|\bar{z}(0)\|\right)$$

Combining Observers & Controllers (continued)

Remember that

$$\bar{z} = \begin{bmatrix} \tilde{x} \\ s \\ e_1 \\ \tilde{p} \end{bmatrix} = \begin{bmatrix} \tilde{x} \\ \dot{\hat{x}} + \tilde{x} \\ x_d - \hat{x} \\ \dot{x}_d - k_{01}\hat{x} + k_{c1}e_1 - (\dot{\hat{x}} - k_{01}\tilde{x}) \end{bmatrix} = \begin{bmatrix} x - \hat{x} \\ \dot{x} - \dot{\hat{x}} + (x - \hat{x}) \\ x_d - \hat{x} \\ \dot{x}_d - k_{01}\tilde{x} + k_{c1}(x_d - \hat{x}) - \left(\dot{\hat{x}} - k_{01}\left(x - \hat{x}\right)\right) \end{bmatrix}$$

Finally, we have semi-global exponential stability, and we can say

$$\underbrace{x \to \hat{x},\ \hat{x} \to x_d,\ \dot{\hat{x}} \to \dot{x},\ \dot{\hat{x}} \to \dot{x}_d}_{\text{so } x \to x_d,\ \dot{x} \to \dot{x}_d} \quad \Rightarrow \text{Occurs exponentially fast!}$$

Recall that you can't measure $\rho_6(\cdot)$, which came from making $f(x,\dot{x}) \in c^1$. Using the Mean Value Theorem and the fact that something is c^1 tells us that

$$\|h(x) - h(\hat{x})\| \le \rho\left(\|\hat{x}\|, \|x\|\right)\|\tilde{x}\|$$

For example,

$$\left\|x^2 - \hat{x}^2\right\| = \|(\hat{x} + x)\tilde{x}\| \le \left(\|\hat{x}\| + \|x\|\right)\|\tilde{x}\|$$

Filter Based Control

Assume we have the same system:

$$\ddot{x} = f(x,\dot{x}) + u$$

where only x is measureable, and the structure of $f(x,\dot{x})$ is uncertain.

We will assume that

$$\rho(x,\dot{x}) \ge \|f(x,\dot{x})\| \quad \Big\} \ \rho(\cdot) \text{ is a positive scalar function}$$

Why couldn't we use this function in the control (if we know ρ)?

It depends on $\dot{x}$ (which we don't know)!

Example:

$$\underbrace{\left\|\dot{x}^2 \cos(x) a\right\|}_{f(x,\dot{x})} \le \underbrace{\bar{a}\left\|\dot{x}^2\right\|}_{\rho(x,\dot{x})} \quad \Big\} \ \text{In the analysis we use the fact that } \rho(\cdot) \text{ exists}$$

The inequality is true, but $\rho(\cdot)$ depends on $\dot{x}$

Filtering Control (continued)

Let's define the following:

$\ddot{e} = \ddot{x}_d - f(x,\dot{x}) - u \quad \Rightarrow$ Why the 2nd derivative? We need it for the control.

We will need e and $\dot{e}$, but $\dot{e}$ is not measureable. So, we come up with another variable:

$\eta = \dot{e} + e + e_f \quad \Rightarrow$ a filter to help us with the $\dot{e}$ problem

We now have three error systems:

error system 1) $\quad \dot{\eta} = \ddot{e} + \dot{e} + \dot{e}_f = \ddot{x}_d - f(x,\dot{x}) + \dot{e} + \dot{e}_f - u$

error system 2) $\quad \dot{e}_f = -e_f - k\eta + e \quad \Rightarrow$ This is designed

error system 3) $\quad \dot{e} = -e_f + \eta - e$ } From the definition of $\eta(t)$

Our next step is to develop a Lyapunov candidate.

$$V = \tfrac{1}{2}e^2 + \tfrac{1}{2}e_f^{\,2} + \tfrac{1}{2}\eta^2$$

Since η is not measureable (due to the fact that $\dot{e}$ is not measureable), we cannot use it in the control. Later we show that e_f is measurable.

Filtering Control (continued)

Taking the derivative of our Lyapunov candidate gives

$$\dot{V} = e(-e + \eta - e_f) + e_f(-e_f - k\eta + e) + \eta(\ddot{x}_d - f(x,\dot{x}) - u)$$
$$+ \eta(-e + \eta - e_f) + \eta(-e_f - k\eta + e)$$
$$\dot{V} = -e^2 - e_f^{\,2} - (k-1)\eta^2 + \eta\left(\ddot{x}_d - f(x,\dot{x}) - u\right) + \eta\left(-(k+2)e_f + e\right)$$

Is e_f measureable?

$$\dot{e}_f = -e_f - k\eta + e = -e_f - k\left(\dot{e} + e + e_f\right) + e = -\left(k+1\right)e_f - \left(k-1\right)e - k\dot{e}$$

Let's develop a new variable p, where

$\dot{p} = function_1$

$e_f = p + function_2$

We will need to find $function_1$ and $function_2$. Differentiate e_f to get

$\dot{e}_f = \dot{p} + \dot{function}_2 = function_1 + \dot{function}_2$

So, that means that

$function_1 = -(k+1)e_f - (k-1)e$

$function_2 = -ke$

Filtering Control (continued)

Now, we know

$\dot{p} = -(k+1)e_f - (k-1)e$ and $e_f = p - ke$

So, e_f is measureable, which leaves $f(x,\dot{x})$ and η as the unmeasureable variables.

We design the control:

$u = \ddot{x}_d - (k+2)e_f + e,$ let $k = 2 + k_n$

Now,

$\dot{V} = -e^2 - e_f{}^2 - \eta^2 + \eta\left(f(x,\dot{x}) - k_n\eta \right)$

Also, we define

$\tilde{f} = \underbrace{f(x_d, \dot{x}_d)}_{f_d} - f(x,\dot{x})$ } Note $\left\| f_d \right\| \le \zeta_d$

If $f \in c^1$, then $\left\|\tilde{f}\right\| \le \rho_1(x_d, \dot{x}_d, e, \dot{e})\left\|\begin{matrix} e \\ \dot{e}\end{matrix}\right\|$ } By the Mean Value Theorem, since $f \in c^1$

$\left\|\tilde{f}\right\| \le \rho_2\left(\|e\|, \|\dot{e}\|\right)\left\|\begin{matrix} e \\ \dot{e}\end{matrix}\right\|$

Filtering Control (continued)

From the previous slide:

$\left\|\tilde{f}\right\| \le \rho_2\left(\|e\|, \|\dot{e}\|\right)\left\|\begin{matrix} e \\ \dot{e}\end{matrix}\right\|$

Let's come up with a new variable z, where

$\|z\| = \left\|\begin{matrix} e \\ e_f \\ \eta \end{matrix}\right\|$ (because $\dot{e} = -e_f + \eta - e$)

So, we know

$\left\|\tilde{f}\right\| \le \rho_3\left(\|z\|\right)\|z\|$

The derivative of our Lyapunov function then becomes

$\dot{V} = -z^T z + \eta\left(-\tilde{f} + f_d - k_n\eta\right) \le -z^T z + \|\eta\|\rho_3\left(\|z\|\right)\|z\| + \|f_d\|\|\eta\| - k_n\|\eta\|^2$

Letting $k_n = k_{n1} + k_{n2}$ and $\|f_d\| \le \zeta_d$ allows us to write

$\dot{V} \le -z^T z + \left(\|\eta\|\rho_3\|z\| - k_{n1}\|\eta\|^2\right) + \left(\|\eta\|\zeta_d - k_{n2}\|\eta\|^2\right)$

Filtering Control (continued)

As seen on the previous slide,

$$\dot{V} \le -z^T z + \left(\|\eta\|\rho_3\|z\| - k_{n1}\|\eta\|^2\right) + \left(\|\eta\|\zeta_d - k_{n2}\|\eta\|^2\right)$$

Now, we can say

$$\dot{V} \le -z^T z + \frac{\|z\|^2 \rho_3{}^2(\|z\|)}{k_{n1}} + \frac{\zeta_d{}^2}{k_{n2}}$$

$$\dot{V} \le -\left(1 - \frac{\rho_3{}^2(\|z\|)}{k_{n1}}\right)\|z\|^2 + \frac{\zeta_d{}^2}{k_{n2}}$$

$$\dot{V} \le -\left(1 - \frac{\rho_3{}^2(\|z\|)}{k_{n1}}\right)\|z\|^2 + \varepsilon, \text{ where } \varepsilon = \frac{\zeta_d{}^2}{k_{n2}}$$

So, we can write

$$\dot{V} \le -\beta\|z\|^2 + \varepsilon, \text{ if } k_{n1} > \rho_3{}^2(\|z\|)$$

$$\dot{V} \le -2\beta V + \varepsilon, \text{ if } k_{n1} > \rho_3{}^2\left(\sqrt{2V(t)}\right)$$

$$\dot{V} + 2\beta V - \varepsilon = s(t) \Rightarrow s(t) \le 0$$

Filtering Control (continued)

Continuing from the previous slide:

$$\dot{V} + 2\beta V - \varepsilon = s(t) \Rightarrow s(t) \le 0$$

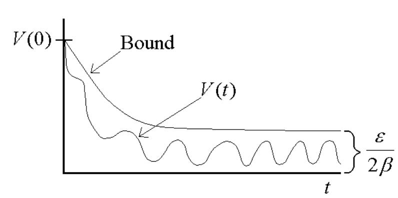
Solving the differential equation gives

$$V(t) = \exp(-2\beta t)V(0) + \varepsilon \exp(-2\beta t)\underbrace{\int_0^t \exp(-2\beta\tau)d\tau}_{\frac{1}{2\beta}(\exp(2\beta t-1)} + \exp(-2\beta t)\int_0^t s(\tau)\exp(2\beta\tau)d\tau$$

$$V(t) \le \exp(-2\beta t)V(0) + \frac{\varepsilon}{2\beta}\left(1 - \exp(-2\beta t)\right)$$

So, $V(t)$ is bounded such that

$$V(t) \le V(0) + \frac{\varepsilon}{2\beta} \Rightarrow$$



Filtering Control (continued)

Continuing from the previous slide:

$$V(t) \le \exp(-2\beta t)V(0) + \frac{\varepsilon}{2\beta}\left(1 - \exp(-2\beta t)\right)$$

We can then write

$$V(t) \le \exp(-2\beta t)V(0) + \frac{\varepsilon}{2\beta}\left(1 - \exp(-2\beta t)\right) \text{ if } k_{n1} \ge \rho_3^{\,2} \left(\sqrt{2\left(V(0) + \frac{\varepsilon}{2\beta}\right)}\right)$$

which means

$$\|z(t)\| \le \sqrt{\|z(0)\|^2 \exp(-2\beta t) + \frac{\varepsilon}{\beta}\left(1 - \exp(-2\beta t)\right)} \text{ if } k_{n1} \ge \rho_3^{\,2}\left(\sqrt{\|z(0)\|^2 + \frac{\varepsilon}{\beta}}\right)$$

So, we have semi-global ultimate uniform boundness. We can easily show that all signals are bounded.

Adaptive Approach

Reconsider the previous system:

$$\ddot{x} = f(x,\dot{x}) + u$$
$$\ddot{e} = -f(x,\dot{x}) - u + \ddot{x}_d$$
$$\eta = \dot{e} + e + e_f$$
$$\dot{e}_f = -e_f - k\eta + e$$
$$\dot{e} = -e_f + \eta - e$$
$$\dot{\eta} = \ddot{e} + \dot{e} + \dot{e}_f = -(k-1)\eta + \ddot{x}_d - f(x,\dot{x}) - u - 2e_f$$

Let

$$u = \ddot{x}_d - 2e_f + e + u_{ff} - ke_f$$

where u_{ff} is a feed forward term, which was not included in our previous control.

This gives

$$\dot{\eta} = -(k-1)\eta - e_f - f(x,\dot{x}) - u_{ff} + ke_f$$

Adaptive Approach (continued)

Consider the Lyapunov candidate

$V = \frac{1}{2}\|z\|^2$ (where $z = [e \;\; e_f \;\; \eta]^T$)

which gives

$\dot{V} = -e^2 - e_f^{\,2} - (k-1)\eta^2 + \eta\left(-f(x,\dot{x}) - u_{ff}\right)$

Assume $f(x,\dot{x}) = W(x,\dot{x})\Theta \;\; \Rightarrow \;\;$ Assume LP

We now write

$$L = \eta\left(-f(x,\dot{x}) - u_{ff}\right) = \eta\left(-f(x,\dot{x}) \underbrace{+f(x_d,\dot{x}_d) - f(x_d,\dot{x}_d)}_{=0} - u_{ff}\right)$$

$$L = \eta\tilde{f} + \eta\left(\underbrace{-W(x_d,\dot{x}_d)\Theta}_{f(x_d,\dot{x}_d)} - u_{ff}\right) \;\; \Rightarrow \;\; \text{Recall that } \tilde{f} = f(x,\dot{x}) - f(x_d,\dot{x}_d)$$

If we can show that $\|\tilde{f}\| \le \rho(\|z\|)\;\|z\|$, and we let $u_{ff} = W(x_d,\dot{x}_d)\hat{\Theta}$, then

$L = \eta\tilde{f} + \eta W(x_d,\dot{x}_d)\tilde{\Theta}$, where $\tilde{\Theta} = \Theta - \hat{\Theta}$

Adaptive Approach (continued)

Now, consider the Lyapunov candidate:

$$V = \frac{1}{2}z^T z + \frac{1}{2}\tilde{\Theta}^T\tilde{\Theta}, \qquad \|z\| = \left\|\begin{matrix} e \\ e_f \\ \eta \end{matrix}\right\| \qquad \tilde{\Theta} = \Theta - \hat{\Theta}$$

where

$\dot{V} = -e^2 - e_f^{\,2} - (k-1)\eta^2 - \eta\tilde{f} - \eta W(x_d,\dot{x}_d)\tilde{\Theta} - \tilde{\Theta}\dot{\hat{\Theta}}$

Our system can now be written

$\ddot{x} = W(x,\dot{x})\Theta - u$, where we assume that $W(x,\dot{x})\;\Theta \in c^1$

$u = \ddot{x}_d - 2e_f + e - ke_f + W(x_d,\dot{x}_d)\hat{\Theta}$

We know that $\|\tilde{f}\| \le \rho(x_d,\dot{x}_d,z)\;\|z\|$ is true since

$\tilde{f} = W(x,\dot{x})\Theta - W(x_d,\dot{x}_d)\Theta$ and $W(x,\dot{x})\;\Theta \in c^1$

Let's create a variable, p, where

$\dot{p} = -(k+1)e_f - (k-1)e$ and $e_f = p - ke$

Let $\dot{e}_f = -e_f - k\eta + e$

Adaptive Approach (continued)

If we let $k = k_n + 2$, then

$$\dot{V} \le -z^T z + \|\eta\| \; |\rho(\cdot)| \; \|z\| - k_n \|\eta\|^2 + \tilde{\Theta}\left(-\eta W(x_d, \dot{x}_d) - \dot{\hat{\Theta}}\right)$$

Where we let $\dot{\hat{\Theta}} = -\eta W(x_d, \dot{x}_d) \quad \Rightarrow \quad \eta$ is NOT measureable! } We address this below

We can know say

$$\dot{V} \le -\|z\|^2 + \frac{\rho^2(\cdot)\|z\|^2}{k_n}$$

$$\dot{V} \le -\left(1 - \frac{\rho^2(\cdot)}{k_n}\right) \|z\|^2$$

We need to use integration by parts:

$$\hat{\Theta} = -\int_0^t \overbrace{W(x_d, \dot{x}_d)(e + e_f}^{measurable} + \underbrace{\dot{e}}_{unmeasurable})d\sigma, \text{ where } \sigma \text{ is just a dummy variable}$$

$$L_1 = -\int_0^t W(x_d, \dot{x}_d)\left(\frac{de}{dt}\right)d\sigma = W(x_d, \dot{x}_d)e\,|_0^t - \int_0^t \dot{W}(x_d, \dot{x}_d)e\,d\sigma$$ } Unmeasurable part

Adaptive Approach (continued)

As seen on the previous slide:

$$L_1 = W(x_d, \dot{x}_d)e\,|_0^t - \int_0^t \dot{W}(x_d, \dot{x}_d)e\,d\sigma$$

Finally,

$$L_1 = W(x_d, \dot{x}_d)e - W\left(x_d(0), \dot{x}_d(0)\right)e(0) - \int_0^t \frac{dW\left(\frac{dx_d}{d\sigma}, x_d(\sigma)\right)}{d\sigma} e(\sigma)\,d\sigma$$

The apadtive update law can now be completed and then we can say

$$\dot{V} \le -\beta\|z\|^2 \text{ if } k_n \ge \rho^2\left(\|z(t)\|\right)$$

Our result is semi-global asymptotic. Why is it not exponential? V has more terms in it than just z.

We can also write

$$\dot{V} \le -\beta\|z\|^2 \text{ for } k_n > \rho^2\left(\sqrt{2V(t)}\right)$$

$$\sqrt{2V - \tilde{\Theta}^T\tilde{\Theta}} = \|z\|$$

$$\dot{V} \le -\beta\|z\|^2 \text{ for } k_n > \rho^2\left(\sqrt{2V(0)}\right)$$

Adaptive Approach (continued)

As seen on the previous slide:

$\dot{V} \le -\beta \|z\|^2$ for $k_n > \rho^2\left(\sqrt{2V(0)}\right)$

It can be shown that $z, \tilde{\Theta} \in L_\infty \Rightarrow e, \dot{e}, e_f, \dot{e}_f, \eta, \dot{\eta}, \hat{\Theta} \in L_\infty$. Why do we care if $\dot{z} \in L_\infty$? We want $\dot{z}(t) \in L_\infty$, which would mean $\lim_{t\to\infty} z(t) = 0$. Remember, z has e, e_f, and η in it. So, they go to zero also. This has been an example of output feedback adaptive control. It gave us semi-global asymptotic tracking.

Why didn't we use an observer (we used a filter)? We don't have exact model knowledge (there is uncertainty in the model)!

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Variable Structure Observer

Consider the system:

$\ddot{x} = h(x,\dot{x}) + G(x,\dot{x})u$, where we observe $\dot{x}$ with only measurements of x.

We also make the assumption that $x, \dot{x}, \ddot{x}, u, \dot{u}, h(x,\dot{x}), G(x,\dot{x}) \in L_\infty$, where $h(x,\dot{x})$, $G(x,\dot{x})$ $\in C^1$ and are uncertain. Why do we make the assumption about boundness? We want to build a $\dot{\hat{x}}$, so we want to ensure that $\dot{\hat{x}} \to \dot{x}$.

For our problem, we define

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}}$$
$$\tilde{x} = x - \hat{x}$$

Let $\dot{\hat{x}} = p + k_o\tilde{x}$, where $\dot{p} = k_1 \operatorname{sgn}(\tilde{x}) + k_2\tilde{x}$ } Observer

Then, $\ddot{\hat{x}} = k_1 \operatorname{sgn}(\tilde{x}) + k_2\tilde{x} + k_o\dot{\tilde{x}}$

$\ddot{\tilde{x}} = h(x,\dot{x}) + G(x,\dot{x})u - k_1 \operatorname{sgn}(\tilde{x}) - k_2\tilde{x} - k_o\dot{\tilde{x}}$ } Observation error system

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Variable Structure Observer (continued)

Let's create a new variable, r, where

$$r = \dot{\tilde{x}} + \tilde{x}$$

$$\dot{r} = h(x,\dot{x}) + G(x,\dot{x})u - k_1 \operatorname{sgn}(\tilde{x}) - k_2\tilde{x} - \left(k_o - I\right)\dot{\tilde{x}}$$

Let $k_2 = k_o - I$. $((k_0)_{ij} > 1; \forall\ i = j)$ Now, we can write

$$\dot{r} = N_o(x,\dot{x},t) - k_1 \operatorname{sgn}(\tilde{x}) - k_2 r$$

So, we have

$$N_o(\cdot) = h(x,\dot{x}) + G(x,\dot{x})u$$

We can let our Lyapunov function be

$$V_o(\cdot) = \tfrac{1}{2} r^T r + P_o(t),\ \text{where } P_o(t) = \zeta_{bo} - \int_{t_0}^{t} L_o(\tau)\,d\tau \quad \Rightarrow \quad \text{we must prove that } P_o(t) \ge 0$$

$$L_o(t) = r^T\left(N_o - k_1 \operatorname{sgn}(\tilde{x})\right)$$

So, we can now write

$$V_o = r^T\left(N_o - k_2 r - k_1 \operatorname{sgn}(\tilde{x})\right) \underbrace{- r^T\left(N_o - k_1 \operatorname{sgn}(\tilde{x})\right)}_{\dot{P}_o = -L_o(t)}$$

Variable Structure Observer (continued)

From the previous slide:

$$V_o = r^T\left(N_o - k_2 r - k_1 \operatorname{sgn}(\tilde{x})\right) - r^T\left(N_o - k_1 \operatorname{sgn}(\tilde{x})\right)$$

Next, we get

$$\dot{V}_o = -r^T k_2 r$$

Using the Rayleigh-Ritz Theorem lets us write

$$\dot{V}_o \le -\lambda_{\min}\{k_2\}\|r\|^2$$

So, $V_o \ge 0$ and $\dot{V}_o \le -g(t)$, where $g(t) \ge 0$. If $\dot{g}(t) \in L_\infty$, then $\lim_{t\to\infty} g(t) = 0$.

Here, $g(t) = \lambda_{\min}\{k_2\} r^T r$ and $\dot{g}(t) = \lambda_{\min}\{k_2\} 2 r^T \dot{r}$

Therefore, $r \in L_\infty \Rightarrow \dot{r} \in L_\infty$, then $r \to 0 \Rightarrow \tilde{x}, \dot{\tilde{x}} \to 0$!

But, we must show that $P_o(t) \ge 0$, which requires

$k_{1i} > |N_{oi}| + |\dot{N}_{oi}|$, where i denotes the ith component for vectors

Variable Structure Observer (continued)

So, our task then is to prove that

$$\zeta_{bo} \geq \int_{t_0}^{t} L_o(\tau)\, d\tau$$

Useful Math Notes:

$$\int_{t_0}^{t} \frac{|y(\tau)|}{y(\tau)} \frac{d(y(\tau))}{d\tau} d\tau = \int_{t_0}^{t} \frac{|y(\tau)|}{y(\tau)} d(y(\tau))$$

$$\sqrt{y^2(\tau)}\,\Big|_{t_0}^{t} = |y(t)| - |y(t_0)| \text{ and } \frac{\partial \sqrt{y^2(t)}}{\partial y} = \frac{y(t)}{|y(t)|} = \frac{|y(t)|}{y(t)}$$

Let $M = \int_{t_0}^{t} L_o(\tau)\, d\tau,$ so we get

$$M = \int_{t_0}^{t} \left[\frac{d\tilde{x}(\tau)}{d\tau}\right]^T \left(N_o - k_1 \operatorname{sgn}(\tilde{x}(\tau))\right) d\tau + \int_{t_0}^{t} \left[\tilde{x}(\tau)\right]^T \left(N_o - k_1 \operatorname{sgn}(\tilde{x}(\tau))\right) d\tau$$

$$M = \int_{t_0}^{t} \left[\frac{d\tilde{x}(\tau)}{d\tau}\right]^T N_o d\tau - \int_{t_0}^{t} \left[\frac{d\tilde{x}(\tau)}{d\tau}\right]^T k_1 \operatorname{sgn}(\tilde{x}(\tau))\, d\tau + \int_{t_0}^{t} \left[\tilde{x}(\tau)\right]^T \left(N_o - k_1 \operatorname{sgn}(\tilde{x}(\tau))\right) d\tau$$

$$M = \left[\tilde{x}(t)\right]^T N_o(\tau)\,\Big|_{t_0}^{t} - \int_{t_0}^{t} \left[\tilde{x}(\tau)\right]^T \frac{d\left(N_o(\tau)\right)}{d\tau} d\tau - \sum_{i=1}^{n} k_{1i} \left|x_i(\tau)\right|\Big|_{t_0}^{t} \ldots$$

$$\ldots + \int_{t_0}^{t} \left[\tilde{x}(\tau)\right]^T \left(N_o - k_1 \operatorname{sgn}(\tilde{x}(\tau))\right) d\tau$$

Variable Structure Observer (continued)

Continuing from the previous slide:

$$M = \int_{t_0}^{t} \left[\tilde{x}(\tau)\right]^T \left(N_o(\tau) - \frac{d\left(N_o(\tau)\right)}{d\tau} - k_1 \operatorname{sgn}(\tilde{x}(\tau)) \right) d\tau + \tilde{x}(t)^T N_o(t) - \tilde{x}(t_0)^T N_o(t_0) \ldots$$

$$\ldots - \sum_{i=1}^{n} k_{1i} \left|\tilde{x}_i(t)\right| + \sum_{i=1}^{n} k_{1i} \left|\tilde{x}_i(t)\right|$$

$$M \leq \int_{t_0}^{t} \left(\sum_{i=1}^{n} k_{1i} \left|\tilde{x}_i(\tau)\right| \left[\left(\left|N_{oi}(\tau)\right| + \left|\frac{d\left(N_{oi}(t)\right)}{dt}\right| \right) - k_{1i} \right] \right) d\tau + \tilde{x}(t)^T N_o(t) \ldots$$

$$\ldots - \tilde{x}(t_0)^T N_o(t_0) - \sum_{i=1}^{n} k_{1i} \left|\tilde{x}_i(t)\right| + \sum_{i=1}^{n} k_{1i} \left|\tilde{x}_i(t)\right|$$

The term $\tilde{x}(t)^T N_o(t)$ can be written $\sum_{i=1}^{n} \left|\tilde{x}_i(t)\right| \left|N_{oi}(t)\right|$, which gives

$$M \leq \sum_{i=1}^{n} \left(k_{1i} \left|\tilde{x}_i(t)\right| - \tilde{x}_i(t_0) N_{oi}(t_0) \right)$$

So, if we define $\zeta_{bo} = M$, then $P_o \geq 0$. Notice that u is not in this observer; so, we can't exploit it for a controller!

Filtering Control, Revisited

Let's consider the following system:

$M(x)\ddot{x} + f(x,\dot{x}) = u,\ x$ is measureable

$$\left.\begin{array}{l} M(x),\ f(x,\dot{x}) \in c^2 \\ M(x), \dot{M}(x) \in L_\infty \text{ if } x, \dot{x} \in L_\infty \\ f(x,\dot{x}),\ \dot{f}(x,\dot{x}) \in L_\infty \text{ if } x, \dot{x}, \ddot{x} \in L_\infty \end{array}\right\} \text{Assumptions}$$

Let $e = x_d - x$ and $M(x)$ be such that

$\underline{M}(x) \le M(x) \le \overline{M}(x) \quad \Rightarrow$ upper and lower bounded

Let $u = k_1 \operatorname{sgn}(e + e_f) - (k_2 + 1) r_f + e$

Let our error system be defined by three equations:

$$\left.\begin{array}{l} \text{error system 1) } \dot{e} = -e - r_f + \eta \\ \text{error system 2) } \dot{r}_f = -r_f - (k_2+1)\eta + e - e_f \\ \text{error system 3) } \dot{e}_f = -e_f + r_f \end{array}\right\} \text{Crafted to make the analysis work}$$

Where did η come from? We invented it.

Filtering Control, Revisited (continued)

We define

$$\dot{p} = -r_f - (k_2+1)(e + r_f) + e - e_f \quad \Big)\ r_f = p - (k_2+1)e$$

$$\eta = \dot{e} + e + r_f$$

Design η such that $\dot{\eta} = \ddot{x}_d - \ddot{x} - 2r_f - e_f - k_2\eta$

Then, by multiplying through by $M(x)$ gives

$$M(x)\dot{\eta} = M(x)(\ddot{x}_d - 2r_f - e_f) - k_2 M(x)\eta + f(x,\dot{x}) - u$$

$$M(x)\dot{\eta} = -k_2 M(x)\eta + N(x,\dot{x},t) - M(x)(2r_f + e_f) - u$$

where $N(\cdot) = M(x)\ddot{x}_d + f(x,\dot{x})$

Then, if we add and subtract an N_d $\left(N_d = N(x,\dot{x},t)\big|_{\substack{x = x_d \\ \dot{x} = \dot{x}_d}}\right.$ is bounded apriori)

We get,

$$M(x)\dot{\eta} = -k_2 M(x)\eta + \tilde{N} + N_d - u - \tfrac{1}{2}\dot{M}(x)\eta$$

Remember that $\tilde{N} = N - N_d$. We can now put in our control:

$$M(x)\dot{\eta} = -k_2 M(x)\eta + \tilde{N} + N_d - k_1 \operatorname{sgn}(e + e_f) + (k_2+1)r_f - e - \tfrac{1}{2}\dot{M}(x)\eta$$

where $\underline{\tilde{N}} = N - N_d - M(x)(2r_f + e_f) + \tfrac{1}{2}\dot{M}(x)\eta$

Filtering Control, Revisited (continued)

As seen on the previous slide:

$$\tilde{N} = N - N_d - M(x)(2r_f + e_f) + \tfrac{1}{2}\dot{M}(x)\eta$$

We can show

$$\|\tilde{N}\| \le \rho(\|z\|)\|z\|,\ z = \begin{bmatrix} e \\ e_f \\ r_f \\ \eta \end{bmatrix}$$

Our next step is to use the Lyapunov function.

$$V = \tfrac{1}{2}M(x)\eta^2 + \tfrac{1}{2}e_f{}^2 + \tfrac{1}{2}r_f{}^2 + \tfrac{1}{2}e^2$$

Where taking the derivative yields

$$\dot{V} = \tfrac{1}{2}\dot{M}(x)\eta^2 + \eta M(x)\dot{\eta} + e_f\dot{e}_f + r_f\dot{r}_f + e\dot{e}$$

Filtering Control, Revisited (continued)

Continuing from the previous slide:

$$\dot{V} = \tfrac{1}{2}\dot{M}(x)\eta^2 + \eta M(x)\dot{\eta} + e_f\dot{e}_f + r_f\dot{r}_f + e\dot{e}$$

$$\dot{V} = -e^2 - er_f + e\eta - e_f{}^2 + e_f r_f - r_f{}^2 - r_f(k_2+1)\eta + r_f e - r_f e_f + \tfrac{1}{2}\dot{M}(x)\eta^2 \ldots$$

$$\ldots - \tfrac{1}{2}\dot{M}(x)\eta^2 + \eta(k_2+1)r_f - \eta e + \eta\tilde{N} - \eta k_2 M(x)\eta + \eta N_d - \eta k_1 \operatorname{sgn}(e+e_f)$$

$$\dot{V} \le -e^2 - e_f{}^2 - r_f{}^2 + \|\eta\|\rho(\|z\|)\|z\| - k_2\underline{M}_1\|\eta\|^2 + \eta N_d - \eta k_1 \operatorname{sgn}(e+e_f)$$

where $\underline{M}_1 \le M(x) \le \overline{M}_2(x)$.

Let $k_2 = \frac{1}{\underline{M}_1}(k_n+1)$. Then, we can write

$$\dot{V} \le -\|z\|^2 + \|\eta\|\left(\rho(\|z\|)\|z\| - k_n\|\eta\|\right) + \eta N_d - \eta k_1 \operatorname{sgn}(e+e_f)$$

$$\dot{V} \le -\|z\|^2 + \frac{\rho^2(\|z\|)\|z\|^2}{k_n} + \eta N_d - \eta k_1 \operatorname{sgn}(e+e_f)$$

$$\dot{V} \le -\left(1 - \frac{\rho^2(\|z\|)}{k_n}\right)\|z\|^2 + \eta N_d - \eta k_1 \operatorname{sgn}(e+e_f)$$

Filtering Control, Revisited (continued)

From the previous slide:

$$\dot{V} \le -\left(1 - \frac{\rho^2(\|z\|)}{k_n}\right)\|z\|^2 + \eta N_d - \eta k_1 \operatorname{sgn}(e + e_f)$$

Keep in mind that

$$\min\{\underline{M}_1, 1\}\|z\|^2 \le V \le \max\{\overline{M}_2(x), 1\}\|z\|^2$$

Rewriting $\dot{V}$ gives

$$\dot{V} \le -\left(1 - \frac{\rho^2(\|z\|)}{k_n}\right)\|z\|^2 + L(t)$$

where $L(t) = \eta N_d - \eta k_1 \operatorname{sgn}(e + e_f)$

Let $V_{new} = V + \underbrace{\zeta_b - \int_0^t L(\tau)d\tau}_{P(t)}$, where ζ_b is a constant.

We have $\dot{V}_{new} = \dot{V} - L(t)$; so,

$$\dot{V}_{new} \le -\left(1 - \frac{\rho^2(\|z\|)}{k_n}\right)\|z\|^2$$

Filtering Control, Revisited (continued)

We have

$$P(t) = \zeta_b - \int_0^t \left(\underbrace{(e + \dot{e} + e_f + \dot{e}_f)}_{\eta} \left(N_d - k_1 \operatorname{sgn}(\underbrace{e + e_f}_{\omega(t)}) \right) \right) d\tau$$

$$P(t) = \zeta_b - \int_0^t \left((\dot{\omega} + \omega)\left(N_d - k_1 \operatorname{sgn}(\omega)\right)\right) d\tau \Rightarrow \text{We've done this before. Work is done!}$$

We know from previous results that $P \ge 0$ if $k_1 > |N_d(\cdot)| + |\dot{N}_d(\cdot)|$

Let $\zeta_b = \left(k_{1i}|e(t_0)| - e(t_0)N_d(t_0)\right)$

Now, we have to complete the proof:

$$\underbrace{\frac{\min}{2}\{\underline{M}_1, 1\}}_{\lambda_1}\|y\|^2 \le V_{new} \le \underbrace{\max\{\frac{\overline{M}_2}{2}, 1\}}_{\lambda_2}\|y\|^2, \text{ where } y = \begin{bmatrix} z^T & \sqrt{P} \end{bmatrix}^T$$

We can then say

$$\dot{V}_{new} \le -\beta\|z\|^2 \text{ for } k_n > \rho^2(\|z\|)$$

$$\dot{V}_{new} \le -\beta\|z\|^2 \text{ for } k_n > \rho^2(\|y\|)$$

Filtering Control, Revisited (continued)

Continuing from the previous slide:

$$\dot{V}_{new} \le -\beta \|z\|^2 \text{ for } k_n > \rho^2\left(\sqrt{\frac{V_{new}(t)}{\lambda_1}}\right)$$

$$\dot{V}_{new} \le -\beta \|z\|^2 \text{ for } k_n > \rho^2\left(\sqrt{\frac{V_{new}(0)}{\lambda_1}}\right)$$

$$\dot{V}_{new} \le -\beta \|z\|^2 \text{ for } k_n > \rho^2\left(\sqrt{\frac{\lambda_2}{\lambda_1}}\|y(0)\|\right)$$

So $V > 0$ and $\dot{V} \le -g(t)$, where $g(t) \ge 0$.

Here $\dot{g}(t) = 2\beta z^T \dot{z}$, and we know if $\dot{g}(t) \in L_\infty$, then $\lim_{t\to\infty} g(t) = 0$.

Therefore $z, e, \dot{e}, e_f, r_f, \eta \to 0$.