Chapter 1: Introduction

1 Linear Time-Invariant Systems

Example 1 *: Mass–spring system*

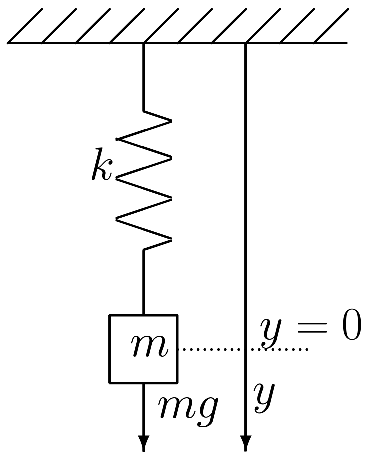

Figure 1: mass–spring system.

$$\begin{aligned} m\ddot{y} &= \sum forces \\ &= f(t) - f_k - f_\beta \end{aligned}$$

$f_\beta = \beta\dot{y}$ friction force; $f_k = ky$ (restoring) spring force. Thus,

$$m\ddot{y} + \beta\dot{y} + ky = mg.$$

Defining states $x_1 = y, x_2 = \dot{y}$, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\beta}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g$$

If our interest is in the displacement y, then

$$y = x_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

thus,

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\beta}{m} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \; D = [0]$$

2 Nonlinear Systems

We are interested in nonlinear systems that can be modeled by a finite number of first-order ordinary differential equations:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \cdots, x_n, t, u_1, \cdots, u_p) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \cdots, x_n, t, u_1, \cdots, u_p) \end{aligned} \tag{1}$$

Defining vectors

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix}, \quad f(t, x, u) = \begin{bmatrix} f_1(x, t, u) \\ \vdots \\ f_n(x, t, u) \end{bmatrix}$$

we will write equation (1) as follows:

$$\dot{x} = f(x, t, u). \tag{2}$$

Similarly, the system output is obtained via the so-called *read out equation*

$$y = h(x, t, u). \tag{3}$$

Special Cases:

- Unforced Systems (u is identically zero).

$$\dot{x} = f(x, t, 0) \ = \ f(x, t). \tag{4}$$

- Autonomous systems ($f(x, t)$ *is not* a function of time).

$$\dot{x} = f(x) \tag{5}$$

Example 2 : *Consider a "hardening spring" where the force strengthens as y increases:*

$$f_k = ky(1 + a^2y^2).$$

in this case we obtain:

$$m\ddot{y} + \beta\dot{y} + ky + ka^2y^3 = f(t).$$

resulting in the following state space realization

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}x_1 - \frac{k}{m}a^2x_1^3 - \frac{\beta}{m}x_2 + \frac{f(t)}{m} \end{cases}$$

which is of the form $\dot{x} = f(x, u)$. In particular, if $u = 0$, then

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}x_1 - \frac{k}{m}a^2x_1^3 - \frac{\beta}{m}x_2 \end{cases}$$

or $\dot{x} = f(x)$. □

Magnetic Suspension System

The equation of the motion of the ball is

$$m\ddot{y} = -f_k + mg + F \tag{6}$$

- m is the mass of the ball,
- f_k is the friction force,
- g the acceleration due to gravity, and
- F is the electromagnetic force due to the current i.

We now look for a model for the magnetic force F. The energy stored in the electromagnet is given by

$$E = \frac{1}{2}Li^2 \qquad L: \text{ Inductance of the electromagnet} \tag{7}$$

$$L = L(y) \approx \frac{\lambda}{1+\mu y}. \tag{8}$$

Thus, $E = E(i, y) = \frac{1}{2}L(y)i^2$, and the force $F = F(i, y)$ is given by

$$F(i, y) = \frac{\partial E}{\partial y} = \frac{i^2}{2}\frac{\partial L(y)}{\partial y} = -\frac{1}{2}\frac{\lambda\mu i^2}{(1+\mu y)^2}. \tag{9}$$

Assuming that the friction force $f_k = k\dot{y}$

$$m\ddot{y} = -k\dot{y} + mg - \frac{1}{2}\frac{\lambda\mu i^2}{(1+\mu y)^2}. \tag{10}$$

Also

$$v = Ri + \frac{d}{dt}(Li) \tag{11}$$

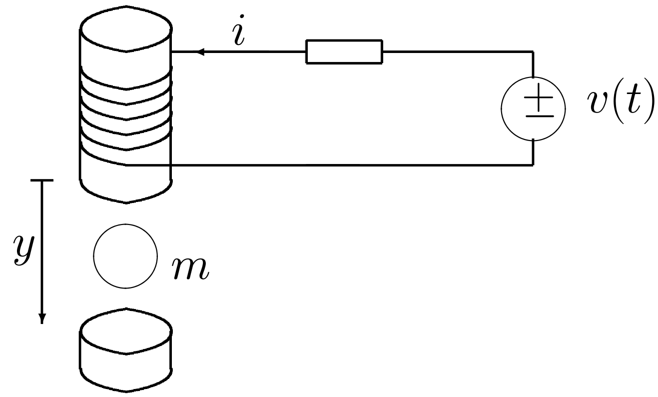

Figure 2: Magnetic suspension system.

where

$$
\begin{aligned}
\frac{d}{dt}(Li) &= \frac{d}{dt}\left(\frac{\lambda i}{1+\mu y}\right) \\
&= \frac{\partial}{\partial y}\left(\frac{\lambda i}{1+\mu y}\right)\ \frac{dy}{dt} + \frac{\partial}{\partial i}\left(\frac{\lambda i}{1+\mu y}\right)\ \frac{di}{dt} \\
&= -\frac{\lambda \mu i}{(1+\mu y)^2}\ \frac{dy}{dt} + \left(\frac{\lambda}{1+\mu y}\right)\ \frac{di}{dt}. \qquad (12)
\end{aligned}
$$

Substituting (12) into (11), we obtain

$$
v = Ri - \frac{\lambda \mu i}{(1+\mu y)^2}\ \frac{dy}{dt} + \frac{\lambda}{1+\mu y}\ \frac{di}{dt}. \qquad (13)
$$

and defining $x_1 = y, x_2 = \dot{y}, x_3 = i$:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= g - \frac{k}{m}x_2 - \frac{\lambda \mu x_3^2}{2m(1+\mu x_1)^2} \\
\dot{x}_3 &= \frac{1+\mu x_1}{\lambda}\ \left[-Rx_3 + \frac{\lambda \mu}{(1+\mu x_1)^2}x_2 x_3 + v\right].
\end{aligned}
$$

Inverted Pendulum on a Cart

$$x = X + \frac{L}{2}\sin\theta = X + l\sin\theta \tag{14}$$
$$y = \frac{l}{2}\cos\theta = l\cos\theta \tag{15}$$

F_x and F_y represent the reaction forces at the pivot point. Consider first the pendulum. Summing forces we obtain the following equations:

$$F_x = m\ddot{X} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta \tag{16}$$
$$F_y - mg = -ml\ddot{\theta}\sin\theta - ml\dot{\theta}^2\cos\theta \tag{17}$$
$$F_y l\sin\theta - F_x l\cos\theta = J\ddot{\theta}. \tag{18}$$

Considering the horizontal forces acting on the cart, we have that

$$M\ddot{X} = f_x - F_x. \tag{19}$$

defining state $x_1 = \theta, x_2 = \dot{\theta}$ we obtain

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{g\sin x_1 - amlx_2^2\sin(2x_1) - 2a\cos(x_1)f_x}{4l/3 - 2aml\cos^2(x_1)}$$

where we have substituted

$$J = \frac{ml^2}{12}, \quad \textit{and} \quad a \triangleq \frac{1}{2(m+M)}$$

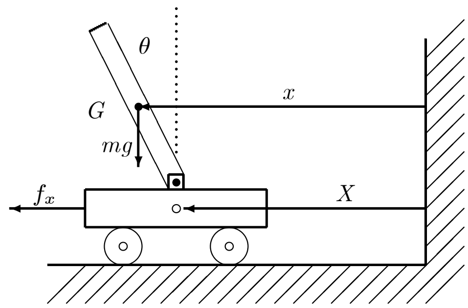

Figure 3: Pendulum-on-a-cart experiment.

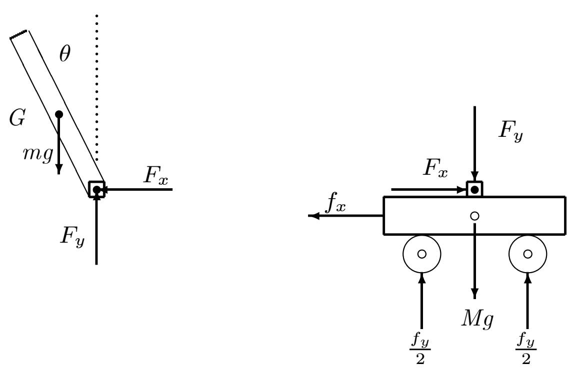

Figure 4: Free-body diagrams of the pendulum-on-a-cart system.

The Ball-and-Beam System

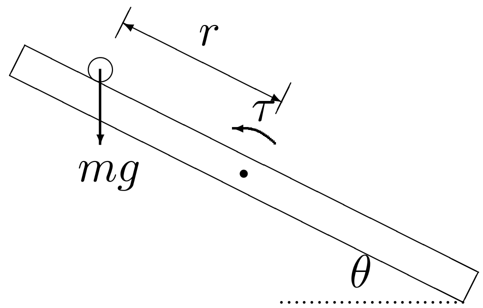

Figure 5: Ball-and-beam experiment.

$$
\begin{aligned}
0 &= (\frac{J_b}{R^2} + m)\ddot{r} + mg\sin\theta - mr\dot{\theta}^2 \\
\tau &= (mr^2 + J + J_b)\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\cos\theta
\end{aligned}
$$

where J: moment of inertia of the beam, and R, m and J_b are the radius, mass and moment of inertia of the ball, respectively. Defining state variables $x_1 = r, x_2 = \dot{r}, x_3 = \theta$, and $x_4 = \dot{\theta}$, we obtain:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{-mg\sin x_3 + mx_1x_4^2}{m + \frac{J_b}{R^2}} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{\tau - mgx_1\cos x_3 - 2mx_1x_2x_4}{mx_1^2 + J + J_b}.
\end{aligned}
$$