Feedback Linearization

1 Mathematical Tools

1.1 Lie Derivative

Definition 1 *: Consider a scalar function $h : D \subset R^n \to R$ and a vector field $f : D \subset R^n \to R^n$. The Lie derivative of h with respect to f, denoted $L_f h$, is given by*

$$L_f h(x) = \frac{\partial h}{\partial x} f(x). \tag{1}$$

Example 1 *: Going back to Lyapunov functions, $\dot{V}$ is the Lie derivative of V with respect to $f(x)$.*

Given two vector fields $f, g : D \subset R^n \to R^n$ we have that

$$L_f h(x) = \frac{\partial h}{\partial x} f(x), \quad L_g h(x) = \frac{\partial h}{\partial x} g(x)$$

and

$$L_g L_f h(x) = L_g[L_f h(x)] = \frac{\partial (L_f h)}{\partial x} g(x)$$

and in the special case $f = g$,

$$L_f L_f h(x) = L_f^2 h(x) = \frac{\partial (L_f h)}{\partial x} f(x).$$

Example 2 *Let*

$$h(x) = \frac{1}{2}(x_1^2 + x_2^2)$$

$$f(x) = \begin{bmatrix} -x_2 \\ -x_1 - \mu(1 - x_1^2)x_2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} -x_1 - x_1 x_2^2 \\ -x_2 + x_1^2 x_2 \end{bmatrix}.$$

Then, we have

- $L_f h(x)$:

$$L_f h(x) = \frac{\partial h}{\partial x} f(x)$$
$$= [x_1 \quad x_2] \begin{bmatrix} -x_2 \\ -x_1 - \mu(1 - x_1^2)x_2 \end{bmatrix}$$
$$= -\mu(1 - x_1^2)x_2^2.$$

- $L_g h(x)$:

$$L_g h(x) = \frac{\partial h}{\partial x} g(x)$$
$$= [x_1 \quad x_2] \begin{bmatrix} -x_1 - x_1 x_2^2 \\ -x_2 + x_1^2 x_2 \end{bmatrix}$$
$$= -(x_1^2 + x_2^2).$$

- $L_f L_g h(x)$:

$$L_f L_g h(x) = \frac{\partial (L_g h)}{\partial x} f(x)$$
$$= -2\,[x_1 \quad x_2] \begin{bmatrix} -x_2 \\ -x_1 - \mu(1 - x_1^2)x_2 \end{bmatrix}$$
$$= 2\mu(1 - x_1^2)x_2^2.$$

1.2 Lie Bracket

Definition 2 *: Consider the vector fields $f, g : D \subset R^n \to R^n$. The Lie bracket of f and g, denoted by $[f, g]$, is the vector field defined by*

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x). \tag{2}$$

Example 3 *: Let*

$$f(x) = \begin{bmatrix} -x_2 \\ -x_1 - \mu(1 - x_1^2)x_2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} [f, g](x) &= \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -x_2 \\ -x_1 - \mu(1 - x_1^2)x_2 \end{bmatrix} \\ &- \begin{bmatrix} 0 & -1 \\ -1 + 2\mu x_1 x_2 & -\mu(1 - x_1^2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -2\mu x_1^2 x_2 \end{bmatrix}. \end{aligned}$$

Notation: (useful when computing repeated bracketing)

$$[f, g](x) \triangleq ad_f g(x)$$

and

$$\begin{aligned} ad_f^0 g &= g \\ ad_f^i g &= [f, ad_f^{i-1} g]. \end{aligned}$$

Thus,

$$\begin{aligned} ad_f^2 g &= [f, ad_f g] = [f, [f, g]] \\ ad_f^3 g &= [f, ad_f^2 g] = [f, [f, [f, g]]]. \end{aligned}$$

1.3 Diffeomorphism

Definition 3 *: (Diffeomorphism) A function $f : D \subset R^n \to R^n$ is said to be a diffeomorphism on D*

(i) it is continuously differentiable on D, and

(ii) its inverse f^{-1} exists and is continuously differentiable.

The function f is said to be a global diffeomorphism if in addition

(i) $D = R^n$, and

(ii) $\lim_{x \to \infty} \|f(x)\| = \infty$.

Lemma 1 *: Let $f(x) : D \in R^n \to R^n$ be continuously differentiable on D. If the Jacobian matrix $Df = \nabla f$ is nonsingular at a point $x_0 \in D$, then $f(x)$ is a diffeomorphism in a subset $\omega \subset D$.*

1.4 Coordinate Transformations

Given

$$\dot{x} = f(x) + g(x)u$$

assuming that $T(x)$ is a diffeomorphism and defining $z = T(x)$, we have that

$$\dot{z} = \frac{\partial T}{\partial x}\dot{x} \;=\; \frac{\partial T}{\partial x}\left[f(x) + g(x)u\right].$$

Since T is a diffeomorphism, we have that $\exists T^{-1}$ and knowing z

$$x = T^{-1}(z).$$

1.5 Distributions

Let $f : D \subset R^n \to R^n$. This vector field assigns the n-dimensional vector $f(x)$ to each point $x \in D$. Now consider "p" vector fields $f_1, f_2, \cdots, f_p$ on $D \subset R^n$. At each $x \in D$

$$\Delta(x) = span\{f_1(x), \cdots, f_p(x)\}$$

is a subspace of R^n.

Definition 4 *: (Distribution) Given an open set $D \subset R^n$ and smooth functions $f_1, f_2, \cdots, f_p : D \to R^n$, we will refer to as a smooth distribution the process of assigning the subspace*

$$\Delta = \{span\{f_1, f_2, \cdots, f_p\}\}$$

spanned by the values of $x \in D$.

Notation:

- $\Delta(x)$: the "values" of Δ at the point x.
- $\{dim\,(\Delta(x)) = \{rank\,([f_1(x), f_2(x), \cdots, f_p(x)])$.
  The dimension of the distribution $\Delta(x)$ at a point $x \in D$ is the dimension of the subspace $\Delta(x)$.

Definition 5 : *Δ is said to be nonsingular if there exists an integer d such that*

$$dim(\Delta(x)) = d \qquad \forall x \in D.$$

otherwise Δ is said to be of variable dimension.

Definition 6 : *A point x_0 of D is said to be a regular point of the distribution Δ if there exist a neighborhood D_0 of x_0 such that Δ is nonsingular on D_0. A point that is not regular is said to be a singularity point.*

Example 4 : *Let $D = \{x \in R^2 : x_1 + x_2 \neq 0\}$ and consider the distribution $\Delta = span\{f_1, f_2\}$, where*

$$f_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 \\ x_1 + x_2 \end{bmatrix}.$$

We have

$$dim(\Delta(x)) = rank\left(\begin{bmatrix} 1 & 1 \\ 0 & x_1 + x_2 \end{bmatrix}\right).$$

Then Δ has dimension 2 everywhere in R^2, except along the line $x_1 + x_2 = 0$. It follows that Δ is nonsingular on D and that every point of D is a regular point.

Example 5 : *Consider the same distribution used in the previous example, but this time with $D = R^2$. From our analysis in the previous example, we have that Δ is not regular since $dim(\Delta(x))$ is not constant over D. Every point on the line $x_1 + x_2 = 0$ is a singular point.*

Definition 7 : *(Involutive Distribution) A distribution Δ is said to be involutive if $g_1 \in \Delta$ and $g_2 \in \Delta \;\Rightarrow\; [g_1, g_2] \in \Delta$.*

It then follows that $\Delta = span\{f_1, f_2, \cdots, f_p\}$ is involutive if and only if

$$rank\,([f_1(x), \cdots, f_p(x)]) \equiv rank\,([f_1(x), \cdots, f_p(x), [f_i, f_j]])\,, \quad \forall x \; andall \; i, j$$

Example 6 *: Let $D = R^3$ and $\Delta = span\{f_1, f_2\}$ where*

$$f_1 = \begin{bmatrix} 1 \\ 0 \\ x_1^2 \end{bmatrix} f_2 = \begin{bmatrix} 0 \\ x_1 \\ 1 \end{bmatrix}.$$

Then it can be verified that $dim(\Delta(x)) = 2\ \forall x \in D$. We also have that

$$[f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore Δ is involutive if and only if

$$rank\left(\begin{bmatrix} 1 & 0 \\ 0 & x_1 \\ x_1^2 & 1 \end{bmatrix}\right) = rank\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & x_1 & 1 \\ x_1^2 & 1 & 0 \end{bmatrix}\right).$$

This, however is not the case, since

$$rank\left(\begin{bmatrix} 1 & 0 \\ 0 & x_1 \\ x_1^2 & 1 \end{bmatrix}\right) = 2 \text{ and } rank\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & x_1 & 1 \\ x_1^2 & 1 & 0 \end{bmatrix}\right) = 3$$

and hence Δ is not involutive.

2 Input–State Linearization

Consider a dynamical systems of the form

$$\dot{x} = f(x) + g(x)u, \quad u \in R$$

Want transform this system into one that is linear time-invariant by using a state feedback control law plus a coordinate transformation.

2.1 Case 1: Systems of the Form $\dot{x} = Ax + B\omega(x)[u - \phi(x)]$

First consider a nonlinear system of the form

$$\dot{x} = Ax + B\omega(x)[u - \phi(x)] \tag{3}$$

where $\omega \neq 0$ in a neighborhood of $x = 0$ and (A, B) controllable. It is immediate that

$$u = \phi(x) + \omega^{-1}(x)v \tag{4}$$

renders the linear time-invariant and controllable system

$$\dot{x} = Ax + Bv$$

Example 7 : *First consider the nonlinear mass–spring system*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}x_1 - \frac{k}{m}a^2x_1^3 - \frac{\omega}{m}x_2 + \frac{f(t)}{m} \end{cases}$$

which can be written in the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & x_2 \\ -\frac{k}{m} & -\frac{\omega}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} [f - ka^2x_1^3].$$

Clearly, this system is of the form (3) with $\omega = 1$ *and* $\phi(x) = ka^2x_1^3$*. It then follows that the linearizing control law is*

$$f = ka^2x_1^3 + v$$

Example 8 : *Now consider the system*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -ax_1 + bx_2 + \cos x_1(u - x_2^2) \end{cases}$$

which is of the form (3) with $\omega = \cos x_1$ *and* $\phi(x) = x_2^2$*. Substituting into (4), we obtain the linearizing control law:*

$$u = x_2^2 + \cos^{-1} x_1 \; v$$

which is well defined for $-\frac{\pi}{2} < x_1 < \frac{\pi}{2}$*.*

2.2 Case 2: Systems of the Form $\dot{x} = f(x) + g(x)u$

Now consider the more general case of affine systems of the form

$$\dot{x} = f(x) + g(x)u. \tag{5}$$

We look for a diffeomorphism $T : D \subset R^n \to R^n$, defining the coordinate transformation

$$z = T(x) \tag{6}$$

and a control law of the form

$$u = \phi(x) + \omega^{-1}(x)v \tag{7}$$

that transform (5) into a state space realization of the form

$$\dot{z} = Az + Bv.$$

Assuming that, after the coordinate transformation (6), the system (5) takes the form

$$\begin{aligned} \dot{z} &= Az + B\bar{\omega}(z)\,[u - \bar{\phi}(z)] \\ &= Az + B\omega(x)\,[u - \phi(x)] \end{aligned} \tag{8}$$

where $\bar{\omega}(z) = \omega(T^{-1}(z))$ and $\bar{\phi}(z) = \phi(T^{-1}(z))$. We have:

$$\dot{z} = \frac{\partial T}{\partial x}\dot{x} = \frac{\partial T}{\partial x}[f(x) + g(x)u]. \tag{9}$$

Substituting (6) and (9) into (8), we have that

$$\frac{\partial T}{\partial x}[f(x) + g(x)u] = AT(x) + B\omega(x)[u - \phi(x)] \tag{10}$$

Equation (10) is satisfied if and only if

$$\frac{\partial T}{\partial x}\, f(x) = AT(x) - B\omega(x)\phi(x) \tag{11}$$

$$\frac{\partial T}{\partial x} g(x) = B\omega(x). \tag{12}$$

Remarks: Assuming (A, B) controllable, we can assume that (A, B) are in the controllable form:

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{n\times n}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}_{n\times 1}$$

Letting

$$T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \\ \vdots \\ T_n(x) \end{bmatrix}_{n\times 1}$$

with $A = A_c,\ B = B_c$ and $z = T(x)$, the right-hand side of equations (11)-(12) becomes

$$A_cT(x) - B_c\omega(x)\phi(x) = \begin{bmatrix} T_2(x) \\ T_3(x) \\ \vdots \\ T_n(x) \\ -\phi(x)\omega(x) \end{bmatrix} \tag{13}$$

and

$$B_c\omega(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \omega(x) \end{bmatrix}. \tag{14}$$

Substituting (13) and (14) into (11) and (12),

$$\begin{aligned} \frac{\partial T_1}{\partial x}f(x) &= T_2(x) \\ \frac{\partial T_2}{\partial x}f(x) &= T_3(x) \\ &\vdots \\ \frac{\partial T_{n-1}}{\partial x}f(x) &= T_n(x) \\ \frac{\partial T_n}{\partial x}f(x) &= -\phi(x)\omega(x) \end{aligned} \tag{15}$$

and

$$\begin{aligned} \frac{\partial T_1}{\partial x} g(x) &= 0 \\ \frac{\partial T_2}{\partial x} g(x) &= 0 \\ &\vdots \\ \frac{\partial T_{n-1}}{\partial x} g(x) &= 0 \\ \frac{\partial T_n}{\partial x} g(x) &= \omega(x) \neq 0. \end{aligned} \tag{16}$$

Thus the components $T_1, T_2, \cdots, T_n$ of the coordinate transformation T must be such that

(i)

$$\begin{aligned} \frac{\partial T_i}{\partial x} g(x) &= 0 \;\; \forall i = 1, 2, \cdots, n-1. \\ \frac{\partial T_n}{\partial x} g(x) &\neq 0. \end{aligned}$$

(ii)

$$\frac{\partial T_i}{\partial x} f(x) = T_{i+1} \qquad i = 1, 2, \cdots, n-1.$$

(iii) The functions ϕ and ω are given by

$$\omega(x) = \frac{\partial T_n}{\partial x} g(x), \quad \phi(x) = -\frac{(\partial T_n/\partial x) f(x)}{(\partial T_n/\partial x) g(x)}.$$

3 Example

Example 9 *: Consider the system*

$$\dot{x} = \begin{bmatrix} e^{x_2} - 1 \\ a x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad = \quad f(x) + g(x)u.$$

We seek a transformation $T = [T_1, T_2]^T$ *such that*

$$\frac{\partial T_1}{\partial x} g \quad = \quad 0 \tag{17}$$

$$\frac{\partial T_2}{\partial x} g \quad \neq \quad 0 \tag{18}$$

with

$$\frac{\partial T_1}{\partial x} f(x) = T_2. \tag{19}$$

In our case, (17) implies that

$$\frac{\partial T_1}{\partial x} g = \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \implies \frac{\partial T_1}{\partial x_2} = 0$$

so that $T_1 = T_1(x_1)$ *(independent of* x_2*). Taking account of (19), we have that*

$$\frac{\partial T_1}{\partial x} f(x) = T_2$$

$$\Rightarrow \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} \end{bmatrix} f(x) \quad = \quad \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & 0 \end{bmatrix} \begin{bmatrix} e^{x_2} - 1 \\ a x_1^2 \end{bmatrix} \quad = \quad T_2$$

$$\Rightarrow T_2 = \frac{\partial T_1}{\partial x_1}(e^{x_2} - 1).$$

To check that (18) is satisfied we notice that

$$\frac{\partial T_2}{\partial x} g(x) \quad = \quad \begin{bmatrix} \frac{\partial T_2}{\partial x_1} & \frac{\partial T_2}{\partial x_2} \end{bmatrix} g(x) \quad = \quad \frac{\partial T_2}{\partial x_2} \quad = \quad \frac{\partial}{\partial x_2} \left(\frac{\partial T_1}{\partial x_1}(e^{x_2} - 1) \right) \neq 0$$

provided that $\frac{\partial T_1}{\partial x_1} \neq 0$*. Thus we can choose*

$$T_1(x) = x_1$$

which results in

$$T = \begin{bmatrix} x_1 \\ e^{x_2} - 1 \end{bmatrix}.$$

The functions ϕ and ω can be obtained as follows:

$$\begin{aligned}\omega &= \frac{\partial T_2}{\partial x} g(x) = e^{x_2} \\ \phi &= -\frac{(\partial T_2/\partial x) f(x)}{(\partial T_2/\partial x) g(x)} = -a x_1^2.\end{aligned}$$

It is easy to verify that, in the z-coordinates

$$\begin{cases} z_1 = x_1 \\ z_2 = e^{x_2} - 1 \end{cases}$$

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = a z_1^2 z_2 + a z_1^2 + (z_2 + 1) u \end{cases}$$

which is of the form

$$\dot{z} = Az + B\omega(z)[u - \phi(z)]$$

with

$$A = A_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\omega(z) = z_2 + 1, \quad \phi(z) = -a z_1^2.$$

4 Conditions for Input–State Linearization

Consider the system

$$\dot{x} = f(x) + g(x)u \tag{20}$$

Theorem 2 *:*
The system (20) is input–state linearizable on $D_0 \subset D$ if and only if the following conditions are satisfied:

(i) The vector fields $\{g(x),\ ad_f g(x), \cdots,\ ad_f^{n-1} g(x)\}$ are linearly independent in D_0.

(ii) The distribution $\Delta = span\{g,\ ad_f g, \cdots,\ ad_f^{n-2} g\}$ is involutive in D_0.

Example 10 *: Consider again the system of Example 9*

$$\dot{x} = \begin{bmatrix} e^{x_2} - 1 \\ a x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad = \quad f(x) + g(x)u.$$

We have,

$$ad_f g = [f, g] = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) \begin{bmatrix} e^{x_2} \\ 0 \end{bmatrix}.$$

Thus, we have that

$$\{g,\ ad_f g\} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} e^{x_2} \\ 0 \end{bmatrix} \right\}$$

and

$$rank(C) = rank\left(\begin{bmatrix} 0 & e^{x_2} \\ 1 & 0 \end{bmatrix} \right) \;=\; 2, \qquad \forall x \in$$

Also the distribution Δ is given by

$$\Delta = span\{g\} = span\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

which is clearly involutive in R^2. Thus conditions (i) and (ii) of Theorem 2 are satisfied $\forall x \in R^2$.

5 Input–Output Linearization

Consider the system

$$\begin{cases} \dot{x} = f(x) + g(x)u & f, g : D \subset R^n \to R^n \\ y = h(x) & h : D \subset R^n \to R \end{cases} \tag{21}$$

We now consider the problem of finding a control law that renders a linear differential equation from the input u to the output y.

Example 11 *: Consider the system of the form*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - a x_1^2 x_2 + (x_2 + 1)u \\ y = x_1. \end{cases}$$

Differentiating the output equation $y = x_1$ we obtain

$$\dot{y} = \dot{x}_1 = x_2$$

which does not contain u. Differentiating once again, we obtain

$$\ddot{y} = \dot{x}_2 = -x_1 - a x_1^2 x_2 + (x_2 + 1)u.$$

Thus, letting

$$u \triangleq \frac{1}{x_2 + 1}\left[v + a x_1^2 x_2\right] \qquad (x_2 \neq -1)$$

we obtain

$$\ddot{y} = -x_1 + v$$

or

$$\ddot{y} + y = v$$

a linear differential equation relating y and the new input v.

We now generalize this idea. Given (21),we proceed as follows.

- Differentiate the output equation to obtain

$$
\begin{aligned}
\dot{y} &= \frac{\partial h}{\partial x}\dot{x} \\
&= \frac{\partial h}{\partial x} f(x) + \frac{\partial h}{\partial x} g(x)u \\
&= L_f h(x) + L_g h(x)u
\end{aligned}
$$

  There are two cases of interest:

  – CASE (1): $L_g h(x) \neq 0 \in D$. In this case we can define

  $$u = \frac{1}{L_g h(x)} \left[-L_f h + v\right]$$

  that renders the linear differential equation

  $$\dot{y} = v.$$

  – CASE (2): $L_g h(x) = 0 \in D$. We continue to differentiate y until u appears explicitly:

  $$\ddot{y} \triangleq y^{(2)} = \frac{d}{dt}\left[\frac{\partial h}{\partial x} f(x)\right] \; = \; L_f^2 h(x) + L_g L_f h(x) u.$$

  We continue to differentiate until, for some integer $r \leq n$

  $$y^{(r)} \; = \; L_f^r h(x) + L_g L_f^{(r-1)} h(x) u$$

  with $L_g L_f^{(r-1)} h(x) \neq 0$. Letting

  $$u = \frac{1}{L_g L_f^{(r-1)} h(x)} \left[-L_f^r h + v\right]$$

  we obtain the linear differential equation

  $$y^{(r)} = v. \tag{22}$$

Definition 8 *: The number of differentiations of y required to obtain (22) is called the relative degree of the system.*

Example 12 : *Consider again the system of Example 11*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - a x_1^2 x_2 + (x_2+1)u \\ y = x_1 \end{cases}$$

we saw that

$$\begin{aligned} \dot{y} &= x_2 \\ \ddot{y} &= -x_1 - a x_1^2 x_2 + (x_2+1)u \end{aligned}$$

hence the system has relative degree 2 in $D_0 = \{x \in R^2 : x_2 \neq -1\}$.

Example 13 : *Consider the linear time-invariant system defined by*

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & -3 & -5 & -1 & -4 \end{bmatrix}_{5\times 5}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}_{5\times 1}$$

$$C = \begin{bmatrix} 7 & 2 & 6 & 0 & 0 \end{bmatrix}_{1\times 5}$$

We have:

$$\begin{aligned} \dot{y} &= CAx + CBu, \qquad CB = 0 \\ y^{(2)} &= CA^2x + CABu, \qquad CAB = 0 \\ y^{(3)} &= CA^3x + CA^2Bu, \qquad CA^2B = 6 \end{aligned}$$

Thus, $r = 3$*. The transfer function associated with this state space realization is*

$$\widehat{H}(s) = C(sI - A)^{-1}B = \frac{6s^2 + 2s + 7}{s^5 + 4s^4 + s^3 + 5s^2 + 3s + 2}$$

thus the relative degree is the excess number of poles over zeros.

6 The Zero Dynamics

We now discuss in more detail the internal dynamics of systems controlled via input–output linearization. Consider first the SISO linear time-invariant system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -q_0 & -q_1 & -q_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} p_0 & p_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The transfer function associated with this system is

$$\widehat{H}(s) = \frac{p_0 + p_1 s}{q_0 + q_1 s + q_2 s^2 + s^3} \; u.$$

Our objective is to design u so that y tracks a desired output y_d. We proceed using the input–output linearization technique (even though the system is linear).

$$\begin{aligned} y &= p_0 x_1 + p_1 x_2 \\ \Rightarrow \dot{y} &= p_0 x_2 + p_1 x_3 \\ \ddot{y} &= p_0 x_3 + p_1(-q_0 x_1 - q_1 x_2 - q_2 x_3) + p_1 u. \end{aligned}$$

Thus, the control law

$$u = \left[q_0 x_1 + q_1 x_2 + q_2 x_3 - \frac{p_0}{p_1} x_3 \right] + \frac{1}{p_1} v$$

produces the simple double integrator

$$\ddot{y} = v.$$

Define the tracking error $e = y - y_d$ and choose $v = -k_1 e - k_2 \dot{e} + \ddot{y}_d$.

$$\begin{aligned} \ddot{e} &= \ddot{y} - \ddot{y}_d \\ \ddot{e} &= -k_1 e - k_2 \dot{e} \end{aligned}$$

With this input v we have that

$$u = \left[q_0 x_1 + q_1 x_2 + q_2 x_3 - \frac{p_0}{p_1} x_3 \right] + \frac{1}{p_1} [-k_1 e - k_2 \dot{e} + \ddot{y}_d]$$

which renders the exponentially stable tracking error closed-loop system

$$\ddot{e} + k_2\dot{e} + k_1 e = 0. \tag{23}$$

Remarks:

- The order of the closed-loop tacking error is the same as the relative order of the system ($r = 2$).
- The original state space realization has order $n = 3$. This means that, after "shifting" the eigenvalues of the A matrix, part of the dynamics of the original system is now unobservable after the input–output linearization.
- To complete the three-dimensional state, we can consider the output equation

  $$y = p_0 x_1 + p_1 x_2 = p_0 x_1 + p_1 \dot{x}_1.$$

  Thus,

  $$\dot{x}_1 = -\frac{p_0}{p_1} x_1 + \frac{1}{p_1} y \quad = \quad A_{id}\, x_1 + B_{id}\, y. \tag{24}$$

- The system (24) contains the unobservable dynamics and is called the internal dynamics. The transfer function of this internal dynamics is

  $$\widehat{H}_{id} = \frac{1}{p_0 + p_1 s}.$$

  which contains a pole whose location in the s plane coincides with that of the zero of $\widehat{H}$, thus leading to the loss of observability.
- This implies that the internal dynamics of the original system is exponentially stable, provided that the zeros of the transfer function $\widehat{H}_{id}$ are in the left-half plane.

The effectiveness of the input-output linearization technique depends upon the stability of the internal dynamics.
Property The zero dynamics can be defined as the internal dynamics of the system when the output is kept identically zero by a suitable input function.

Example 14 : *Consider the system*

$$\begin{cases} \dot{x}_1 = -kx_1 - 2x_2u \\ \dot{x}_2 = -x_2 + x_1u \\ y \;=\; x_2 \end{cases}$$

Differentiating y, *we obtain*

$$\dot{y} = \dot{x}_2 = -x_2 + x_1u.$$

Therefore, $r = 1$. *To determine the zero dynamics, we proceed as follows:*

$$y = 0 \quad \Longleftrightarrow \quad x_2 = 0 \quad \Longleftrightarrow \quad u = 0.$$

Then, the zero dynamics is given by

$$x_1 = -kx_1$$

which is exponentially stable (globally) if $k > 0$, *and unstable if* $k < 0$.

Example 15 : *Consider the system*

$$\begin{cases} \dot{x}_1 = x_2 + x_1^2 \\ \dot{x}_2 = x_2^3 + u \\ \dot{x}_3 = x_1 + x_2^3 + \alpha x_3 \\ y \;= x_1 \end{cases}$$

Differentiating y, *we obtain*

$$\dot{y} = \dot{x}_1 = x_2 + x_1^2$$
$$\ddot{y} = 2x_1\dot{x}_1 + \dot{x}_2 = 2x_1(x_2 + x_1^2) + x_2^3 + u$$

Therefore $r = 2$. *To find the zero dynamics, we proceed as follows:*

$$y = 0 \Longleftrightarrow x_1 = 0 \quad \Rightarrow \dot{x}_1 = 0x_2 + x_1^2 \quad \Rightarrow x_2 = 0$$

$$\dot{x}_2 = x_2^3 + u = 0 \quad \Longleftrightarrow \quad u = -x_2^3.$$

Therefore the zero dynamics is given by

$$\dot{x}_3 = \alpha x_3.$$

Moreover, the zero dynamics is asymptotically stable if $\alpha < 0$, *and unstable if* $\alpha > 0$.