Chapter 11: Nonlinear Observers

1 Nonlinear Observability

Consider the system

$$\psi_{nl} \begin{cases} \dot{x} = f(x) + g(x)u & f : \mathbf{R}^n \to \mathbf{R}^n, g : R^n \to \mathbf{R} \\ y = h(x) & h : \mathbf{R}^n \to \mathbf{R} \end{cases} \tag{1}$$

Notation:

- $x_u(t, x_0)$: solution of (1) at time t originated by u with initial state x_0.
- $y(x_u(t, x_0)$: output y when the state x is $x_u(t, x_0)$.

Clearly

$$y(x_u(t, x_0)) \equiv h(x_u(t, x_0))$$

Definition 1 *: A pair of states (x_0^1, x_0^2) is said to be* distinguishable *if there exists an input function u such that*

$$y(x_u(t, x_0^1)) \equiv y(x_u(t, x_0^2))$$

Definition 2 *: ψ_{nl} is said to be (locally) observable at $x_0 \in \mathbf{R}^n$ if there exists a neighborhood U_0 of x_0 such that every state $x \neq x_0 \in \Omega$ is distinguishable from x_0. It is said to be locally observable if it is locally observable at each $x_0 \in \mathbf{R}^n$.*

This means that ψ_{nl} is locally observable in a neighborhood $U_0 \subset \mathbf{R}^n$ if there exists an input $u \in \mathbf{R}$ such that

$$y(x_u(t, x_0^1)) \equiv y(x_u(t, x_0^2)) \qquad \forall t \in [0, t] \qquad \Longleftrightarrow \qquad x_0^1 = x_0^2$$

Remarks There is no requirement in Definition 2 that distinguishability must hold for <u>all</u> functions.

In the following theorem we consider a system of the form:

$$\psi_{nl} \begin{cases} \dot{x} = f(x) & \qquad f : \mathbf{R}^n \to \mathbf{R}^n \\ y = h(x) & \qquad h : \mathbf{R}^n \to \mathbf{R} \end{cases} \tag{2}$$

Theorem 1 *The state space realization (2) is locally observable in a neighborhood $U_0 \subset D$ containing the origin, if*

$$rank\left(\begin{bmatrix} \nabla h \\ \vdots \\ \nabla L_f^{n-1} h \end{bmatrix}\right) = n \quad \forall x \in U_0 \tag{3}$$

Example 1 *(Linear time-invariant)*

$$\psi_l \begin{cases} \dot{x} = Ax \\ y = Cx. \end{cases}$$

Then $h(x) = Cx$ and $f(x) = Ax$, and we have

$$\begin{aligned} \nabla h(x) &= C \\ \nabla L_f h &= \nabla(\frac{\partial h}{\partial x}\dot{x}) = \nabla(CAx) = CA \\ &\vdots \\ \nabla L_f^{n-1} h &= CA^{n-1} \end{aligned}$$

and therefore ψ_l is observable if and only if $S = \{C, CA, CA^2, \cdots, CA^{n-1}\}$ is linearly independent or, equivalently, if $rank(\mathcal{O}) = n$.

Example 2 *Consider the following state space realization:*

$$\psi_{nl} \begin{cases} \dot{x}_1 = x_2(1-u) \\ \dot{x}_2 = x_1 \\ y \ \ = x_1 \end{cases}$$

which is of the form

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

with

$$f(x) = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}, \quad g(x) = \begin{pmatrix} -x_2 \\ 0 \end{pmatrix}, \quad h(x) = x_1.$$

If $u = 0$, we have

$$rank(\{\nabla h, \nabla L_f h\}) = rank(\{[1 \ \ 0], [0 \ \ 1]\}) = 2$$

and thus

$$\begin{cases} \dot{x} = f(x) \\ y = h(x) \end{cases}$$

is observable according to Definition 2. Assume now that $u = 1$. We obtain the following dynamical equations:

$$\begin{cases} \dot{x}_1 = 0 \\ \dot{x}_2 = x_1 \\ y \ \ = x_1. \end{cases}$$

A glimpse at the new linear time-invariant state space realization shows that observability has been lost.

1.1 Nonlinear Observers

There are several ways to approach the nonlinear state reconstruction problem, depending on the characteristics of the plant. We now discuss two rather different approaches, each applicable to a particular class of systems.

2 Observers with Linear Error Dynamics

Idea:

(i) Find an invertible coordinate transformation that linearizes the state space realization.

(ii) Design an observer for the resulting linear system.

(iii) Recover the original state using the inverse coordinate transformation defined in (i).

More explicitly, suppose that given a system of the form

$$\begin{cases} \dot{x} = f(x) + g(x,u) & x \in \mathbf{R}^n,\ u \in \mathbf{R} \\ y = h(x) & y \in \mathbf{R} \end{cases} \tag{4}$$

there exist a diffeomorphism $T(\cdot)$ satisfying

$$z = T(x), \qquad T(0) = 0, \quad z \in \mathbf{R}^n \tag{5}$$

and such that, after the coordinate transformation, the new state space realization has the form

$$\begin{cases} \dot{z} = A_0 z + \gamma(y,u) & \\ y = C_0 z & y \in \mathbf{R} \end{cases} \tag{6}$$

where

$$A_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & & \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad C_0 = [0 \;\; 0 \;\; \cdots \;\; 0 \;\; 1], \;\; \gamma = \begin{bmatrix} \gamma_1(y,u) \\ \gamma_1(y,u) \\ \vdots \\ \vdots \\ \gamma_n(y,u) \end{bmatrix}. \qquad (7)$$

then, under these conditions, an observer can be constructed according to the following theorem.

Theorem 2 *(Marino-Tomei) If there exist a coordinate transformation mapping (4) into (6), then, defining*

$$\begin{aligned} \dot{\widehat{z}} &= A_0\widehat{z} + \gamma(y,u) - K(y - (C_0\widehat{z}_n)), \qquad \widehat{z} \in \mathbf{R}^n & (8) \\ \widehat{x} &= T^{-1}(\widehat{z}) & (9) \end{aligned}$$

such that the eigenvalues of $(A_0 + KC_0)$ *are in the left half of the complex plane, then* $\widehat{x} \to x$ *as* $t \to \infty$*.*

Proof: Let $\tilde{z} = z - \widehat{z}$, and $\tilde{x} = x - \widehat{x}$. We have

$$\begin{aligned} \dot{\tilde{z}} &= \dot{z} - \dot{\widehat{z}} \\ &= [A_0 z + \gamma(y,u)] - [A_0\widehat{z} + \gamma(y,u) - K(y - \widehat{z}_n)] \\ &= (A_0 + KC_0)\tilde{z}. \end{aligned}$$

If the eigenvalues of $(A_0 + KC_0)$ have negative real part, then we have that $\tilde{z} \to 0$ as $t \to \infty$. Using (9), we obtain

$$\begin{aligned} \tilde{x} &= x - \widehat{x} \\ &= T^{-1}(z) - T^{-1}(\hat{z}) \qquad \to 0 \;\; as \;\; t \to \infty \end{aligned}$$

Example 3 *Consider the following dynamical system*

$$\begin{cases} \dot{x}_1 = x_2 + 2x_1^2 \\ \dot{x}_2 = x_1 x_2 + x_1^3 u \\ y = x_1 \end{cases} \tag{10}$$

and define the coordinate transformation

$$\begin{cases} z_1 = x_2 - \frac{1}{2}x_1^2 \\ z_2 = x_1. \end{cases}$$

In the new coordinates, the system (10) takes the form

$$\begin{cases} \dot{z}_1 = -2y^3 + y^3 u \\ \dot{z}_2 = z_1 + \frac{5}{2}y^2 \\ y = z_2 \end{cases}$$

which is of the form (6) with

$$A_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad C_0 = [0 \;\; 1], \quad \text{and } \gamma = \begin{bmatrix} -2y^3 + y^3 u \\ \frac{5}{2}y^2 \end{bmatrix}.$$

The observer is

$$\dot{\widehat{z}} = A_0 \widehat{z} + \gamma(y,u) - K(y - \widehat{z}_2)$$

$$\begin{bmatrix} \dot{\widehat{z}}_1 \\ \dot{\widehat{z}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \widehat{z}_1 \\ \widehat{z}_2 \end{bmatrix} + \begin{bmatrix} -2y^3 + y^3 u \\ 5y^2/2 \end{bmatrix} + \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} (y - \widehat{z}_2).$$

The error dynamics is

$$\begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & -K_1 \\ 1 & -K_2 \end{bmatrix} \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix}$$

Thus, $\tilde{z} \to 0$ *for any* $K_1, K_2 > 0$.

3 Lipschitz Systems

Consider a system of the form:

$$\begin{cases} \dot{x} = Ax + f(x,u), & A \in \mathbf{R}^{n\times n} \\ y = Cx & C \in R^{1\times n} \end{cases} \tag{11}$$

where f is Lipschitz in x on an open set $D \subset \mathbf{R}^n$, i.e.,

$$\|f(x_1,u^*) - f(x_2,u^*)\| \ \le \ \gamma\|x_1 - x_2\| \quad \forall x \in D. \tag{12}$$

Now consider the following observer structure

$$\dot{\widehat{x}} = A\widehat{x} + f(\widehat{x},u) + L(y - C\widehat{x}) \tag{13}$$

where $L \in \mathbf{R}^{n\times 1}$.

Theorem 3 *Given the system (11) and the corresponding observer (13), if the Lyapunov equation*

$$P(A - LC) + (A - LC)^T P = -Q \tag{14}$$

where $P = P^T > 0$, and $Q = Q^T > 0$, is satisfied with

$$\gamma < \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)} \tag{15}$$

then the observer error $\tilde{x} = x - \widehat{x}$ is asymptotically stable.

Proof:

$$
\begin{aligned}
\dot{\tilde{x}} &= \dot{x} - \dot{\widehat{x}} \\
&= [Ax + f(x,u)] - [A\widehat{x} + f(\widehat{x},u) + L(y - C\widehat{x})] \\
&= (A - LC)\tilde{x} + f(x,u) - f(\widehat{x},u).
\end{aligned}
$$

To see that $\tilde{x}$ has an asymptotically stable equilibrium point at the origin, consider the Lyapunov function candidate:

$$V(\tilde{x}) \triangleq \tilde{x}^T P \tilde{x}.$$

$$
\begin{aligned}
\dot{V}(\tilde{x}) &= \dot{\tilde{x}}^T P\tilde{x} + \tilde{x}^T P\tilde{x}\ \dot{\tilde{x}} \\
&= -\ \dot{\tilde{x}}^T Q\tilde{x} + 2\tilde{x}^T P[f(x,u) - f(\widehat{x},u)]
\end{aligned}
$$

but

$$\|\tilde{x}^T Q\tilde{x}\| \geq \lambda_{min}(Q)\|\tilde{x}\|^2$$

and

$$\|f(x_1,u^*) - f(x_2,u^*)\| \ \leq \ \gamma\|x_1 - x_2\| \quad \forall x \in D$$

so

$$
\begin{aligned}
\dot{V}(\tilde{x}) &\leq -\lambda_{min}(Q)\|\tilde{x}\|^2 + 2\tilde{x}^T P[f(x,u) - f(\widehat{x},u)] \\
\dot{V}(\tilde{x}) &\leq -\lambda_{min}(Q)\|\tilde{x}\|^2 + 2\|\tilde{x}\|\ \|P\|\ \|f(x,u) - f(\widehat{x},u)\| \\
\dot{V}(\tilde{x}) &\leq -\lambda_{min}(Q)\|\tilde{x}\|^2 + 2\|\tilde{x}\|\ \|P\|\ \gamma\|\tilde{x}\| \\
\dot{V}(\tilde{x}) &\leq -\lambda_{min}(Q)\|\tilde{x}\|^2 + 2\gamma\|P\|\ \|\tilde{x}\|^2 \\
\dot{V}(\tilde{x}) &\leq -\{\lambda_{min}(Q) - 2\gamma\|P\|\}\|\tilde{x}\|^2
\end{aligned}
$$

Therefore, $\dot{V}$ is negative definite, provided that

$$\lambda_{min}(Q) > 2\gamma\|P\|$$

or, equivalently

$$\gamma < \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)}$$

Example 4 *Consider the following system:*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix}$$

$$y = [1 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

we have that

$$L = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \;\; \Rightarrow \;\; A - LC = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

Solving the Lyapunov equation

$$P(A - LC) + (A - LC)^T P = -Q$$

with $Q = I$, we obtain

$$P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

which is positive definite. The eigenvalues of P are $\lambda_{min}(P) = 0.2929$, and $\lambda_{max}(P) = 1.7071$. We now consider the function f.

Aside

$$x_1' = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \qquad x_2' = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

we have that

$$\begin{aligned} \|f(x_1') - f(x_2')\|_2 &= \sqrt{(\xi_2^2 - \mu_2^2)^2} \\ &= |\xi_2^2 - \mu_2^2| \\ &= |(\xi_2 + \mu_2)(\xi_2 - \mu_2)| \\ \leq 2|\xi_2| \, |\xi_2 - \mu_2| &= 2k \, \|\xi_2 - \mu_2\| \\ &\leq 2k\|x_1' - x_2'\|_2 \end{aligned}$$

for all x satisfying $|\xi_2| < k$. Thus, $\gamma = 2k$ and f is Lipschitz $\forall x = [\xi_1 \;\; \xi_2]^T :$ $|\xi_2| < k$, and we have

$$\gamma = 2k < \frac{1}{2\lambda_{max}(P)}$$

or

$$k < \frac{1}{6.8284}$$