Chapter 3: Lyapunov Stability I: Autonomous Systems

1 Definitions

Consider the autonomous system

$$\dot{x} = f(x) \qquad\qquad f : D \to R^n \tag{1}$$

Definition 1 *: x_e is an equilibrium point of (1) if*

$$f(x_e) = 0.$$

We want to know whether or not the trajecories near an equilibrium point are "well behaved".

Definition 2 *: x_e is said to be stable if for each $\epsilon > 0$,*

$$\exists \delta = \delta(\epsilon) > 0$$

$$\|x(0) - x_e\| < \delta \quad \Rightarrow \quad \|x(t) - x_e\| < \epsilon \quad \forall t \geq t_0$$

otherwise, the equilibrium point is said to be unstable.

<u>IMPORTANT</u>: This notion applies to the equilibrium, not "the system." A dynamical system can have several equilibrium points.

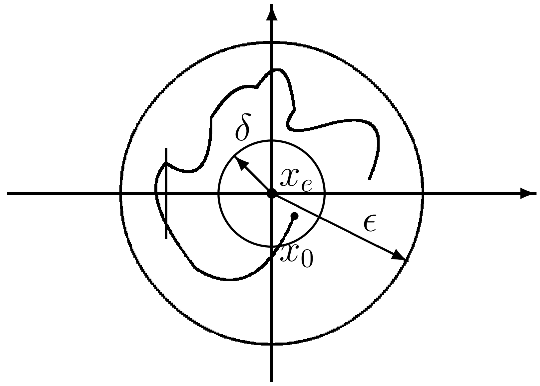

Figure 1: Stable equilibrium point.

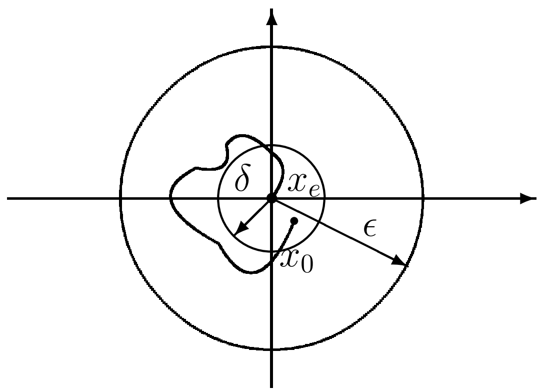

Figure 2: Asymptotically stable equilibrium point.

Definition 3 *: x_e of the system (1) is said to be convergent if there exists $\delta_1 > 0$:*

$$\|x(0) - x_e\| < \delta_1 \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = x_e.$$

Equivalently, x_e is convergent if for any given $\epsilon_1 > 0, \exists T$ such that

$$\|x(0) - x_e\| < \delta_1 \quad \Rightarrow \quad \|x(t) - x_e\| < \epsilon_1 \quad \forall t \geq t_0 + T.$$

Definition 4 *: x_e is said to be asymptotically stable if it is both stable and convergent.*

Definition 5 *: x_e is said to be (locally) exponentially stable if there exist two real constants $\alpha, \lambda > 0$ such that*

$$\|x(t) - x_e\| \leq \alpha \, \|x(0) - x_e\| \, e^{-\lambda t} \qquad \forall t > 0 \tag{2}$$

whenever $\|x(0) - x_e\| < \delta$. It is said to be globally exponentially stable if (2) holds for any $x \in R^n$.

Note: Clearly, exponential stability implies asymptotic stability. The converse is, however, not true.

Remarks: We can always assume that $x_e = 0$. Given any other equilibrium point we can make a change of variables and define a new system with an equilibrium point at $x = 0$. Define:

$$\begin{aligned} y &= x - x_e \\ \Rightarrow \dot{y} &= \dot{x} \;=\; f(x) \\ \Rightarrow f(x) &= f(y + x_e) \;\triangleq\; g(y). \end{aligned}$$

Thus, the equilibrium point y_e of the new systems $\dot{y} = g(y)$ is $y_e = 0$, since

$$g(0) = f(0 + x_e) \;=\; f(x_e) \;=\; 0.$$

Thus, x_e is stable for the system $\dot{x} = f(x)$ if and only if $y = 0$ is stable for the system $\dot{y} = g(y)$.

2 Positive Definite Functions

Definition 6 *:* $V : D \to R$ *is* *positive semi definite* *in* D *if*

(i) $0 \in D$ *and* $V(0) = 0$.

(ii) $V(x) \geq 0, \quad \forall x$ *in* $D - \{0\}$.

$V : D \to R$ *is* *positive definite* *in* D *if*

(ii') $V(x) > 0$ *in* $D - \{0\}$.

$V : D \to R$ *is* *negative definite* *(semi definite)in* D *if* $-V$ *is positive definite (semi definite).*

Example 1 : *(Quadratic form)*

$$V(x) : R^n \to R = x^T Q x \ , \quad Q \in R^{n\times n}, \ \ Q = Q^T.$$

Since $Q = Q^T$, its eigenvalues $\lambda_i, i = 1, \cdots n$, are all real. Thus

$$\begin{aligned}
V(\cdot) \textit{ positive definite } &\Longleftrightarrow \lambda_i > 0, \forall i = 1, \cdots, \ n \\
V(\cdot) \textit{ positive semidefinite } &\Longleftrightarrow \lambda_i \geq 0, \forall i = 1, \cdots, \ n \\
V(\cdot) \textit{ negative definite } &\Longleftrightarrow \lambda_i < 0, \forall i = 1, \cdots, \ n \\
V(\cdot) \textit{ negative semidefinite } &\Longleftrightarrow \lambda_i \leq 0, \forall i = 1, \cdots, \ n
\end{aligned}$$

□

Notation: Given a dynamical system and a function V we will denote

$$\begin{aligned}
\dot{V}(x) &= \frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \nabla V \cdot f(x) \\
&= \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \cdots, \frac{\partial V}{\partial x_n}\right] \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}.
\end{aligned}$$

Example 2 : *Let*

$$\dot{x} = \begin{bmatrix} ax_1 \\ bx_2 + \cos x_1 \end{bmatrix}$$

and define $V = x_1^2 + x_2^2$. Thus, we have

$$\begin{aligned}
\dot{V}(x) = \frac{\partial V}{\partial x} f(x) &= [2x_1, 2x_2] \begin{bmatrix} ax_1 \\ bx_2 + \cos x_1 \end{bmatrix} \\
&= 2ax_1^2 + 2bx_2^2 + 2x_2 \cos x_1.
\end{aligned}$$

□

Notice that the $\dot{V}(x)$ depends on the system's equation $f(x)$ and thus it will be different for different systems.

Note:

$$
\begin{aligned}
V &= x^T Q x, \qquad \text{where} \qquad Q = Q^T \\
\\
\text{if } Q &= Q_{sym} + Q_{asym} \\
Q_{sym} &= \frac{Q + Q^T}{2} \\
Q_{asym} &= \frac{Q - Q^T}{2} \\
\text{Then, } V &= x^T Q_{sym} x + x^T Q_{asym} x \\
\text{where, } Q_{asym} &= \begin{bmatrix} 0 & a & b \\ -a & \ddots & \vdots \\ -b & \cdots & 0 \end{bmatrix} \text{ i.e. skew symmetric} \\
\text{and } x^T Q_{asym} x &= 0 \\
\therefore V &= x^T Q_{sym} x \qquad \therefore \text{Q is always symmetric}
\end{aligned}
$$

Theorem: Rayleigh-Ritz theorem

$$\lambda_{\min}\{Q\}\|x\|^2 \leq x^T Q x \leq \lambda_{\max}\{Q\}\|x\|^2$$

3 Stability Theorems

Theorem 1 *: (Lyapunov Stability Theorem) Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$, and let $V : D \to R$ be a continuously differentiable function such that*

(i) $V(0) = 0$,

(ii) $V(x) > 0 \quad in \;\; D - \{0\}$,

(iii) $\dot{V}(x) \leq 0 \quad in \; D - \{0\}$,

thus $x = 0$ is stable.

Theorem 2 *: (Asymptotic Stability Theorem) Under the conditions of Theorem 1, if $V(\cdot)$ is such that*

(i) $V(0) = 0$,

(ii) $V(x) > 0 \quad in \;\; D - \{0\}$,

(iii) $\dot{V}(x) < 0 \quad in \; D - \{0\}$,

thus $x = 0$ is asymptotically stable.

Proof of theorem 1: Choose $r > 0$ and define

$$B_r = \{x \in R^n : \|x\| \leq r\} \subset D$$

is contained in D. B_r, so defined is a closed and bounded ("compact") set (a sphere).

We now construct a Lyapunov surface inside B_r and show that all trajectories starting near $x = 0$ remain inside the surface. Let

$$\alpha = \min_{\|x\|=r} V(x) \qquad \text{(thus } \alpha > 0\text{)}$$

Now choose $\beta \in (0, \alpha)$ and denote

$$\Omega_\beta = \{x \in B_r : V(x) \leq \beta\}.$$

Thus, by construction, $\Omega_\beta \subset B_r$. Assume now that $x(0) \in \Omega_\beta$.

$$\dot{V}(x) \leq 0 \quad \Rightarrow \quad V(x) \leq V(x(0)) \leq \beta \quad \forall t \geq 0.$$

Trajectories starting in Ω_β at $t = 0$ stays inside Ω_β for all $t \geq 0$. By the continuity of $V(x)$, $\exists \delta > 0$:

$$\|x\| < \delta \Rightarrow V(x) < \beta \qquad (B_\delta \subset \Omega_\beta \subset B_r).$$

It then follows that

$$\|x(0)\| < \delta \Rightarrow x(t) \in \Omega_\beta \subset B_r \quad \forall t > 0$$

and then

$$\|x(0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < r \leq \epsilon \quad \forall t \geq 0$$

which means that the equilibrium $x = 0$ is stable. □

Proof of theorem 2: Similar, only that $\dot{V}(x) < 0 \quad \mathit{in}\ D$ implies that Ω_β is "shrinking" until eventually it becomes the single point $x = 0$. □

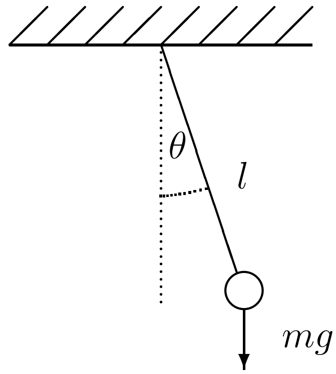

Figure 3: Pendulum without friction.

4 Examples

Example 3 : *(Pendulum Without Friction)*
Using Newton's second law of motion we have,

$$\begin{aligned} ma &= -mg\sin\theta \\ a &= l\alpha = l\ddot{\theta} \end{aligned}$$

where l is the length of the pendulum, and α is the angular acceleration. Thus

$$\begin{aligned} ml\ddot{\theta} + mg\sin\theta &= 0 \\ \textit{or} \quad \ddot{\theta} + \frac{g}{l}\sin\theta &= 0 \end{aligned}$$

choosing state variables

$$\begin{cases} x_1 = \theta \\ x_2 = \dot{\theta} \end{cases}$$

we have

$$\begin{cases} \dot{x}_1 = x_2 = f_1(x) \\ \dot{x}_2 = -\frac{g}{l}\sin x_1 = f_2(x) \end{cases}$$

To study the stability of the equilibrium at the origin, we need to propose a Lyapunov function candidate $V(x)$. This is difficult! In this case we "try" the total energy, which is a positive function. We have

$$\begin{aligned} E &= K + P \qquad \textit{(kinetic plus potential energy)} \\ &= \frac{1}{2}m(\omega l)^2 + mgh \end{aligned}$$

where

$$\begin{aligned}\omega &= \dot{\theta} = x_2 \\ h &= l(1-\cos\theta) = l(1-\cos x_1).\end{aligned}$$

Thus

$$E = \frac{1}{2}ml^2x_2^2 + mgl(1-\cos x_1).$$

We now define $V(x) = E$. *We see that because of the periodicity of* $\cos(x_1)$, *we have that* $V(x) = 0$ *whenever* $x = (x_1, x_2)^T = (2k\pi, 0)^T$, $k = 1, 2, \cdots$. *Thus,* $V(\cdot)$ *is not positive definite. However, restricting the domain of* x_1 *to the interval* $(-2\pi, 2\pi)$; *i.e., we take* $V : D \to R$, *with* $D = ((-2\pi, 2\pi), R)^T$. *We have that* $V : D \to R > 0$ *is indeed positive definite. Also*

$$\begin{aligned}\dot{V}(x) &= \nabla V \cdot f(x) \\ &= \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}\right][f_1(x), f_2(x)]^T \\ &= [mgl\sin x_1, ml^2x_2][x_2, -\frac{g}{l}\sin x_1]^T \\ &= mglx_2\sin x_1 - mglx_2\sin x_1 = 0.\end{aligned}$$

Thus $\dot{V}(x) = 0$ *and the origin is stable by Theorem 1.* □

Example 4 *: (Pendulum with Friction) We now modify the previous example by adding the friction force* $kl\dot{\theta}$

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$$

defining the same state variables as in example 3 we have

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2. \end{cases}$$

Again $x = 0$ *is an equilibrium point. The energy is the same as in Example 3. Thus*

$$V(x) = \frac{1}{2}ml^2x_2^2 + mgl(1-\cos x_1) \quad > 0 \qquad in \;\; D - \{0\}$$

$$\begin{aligned} \dot{V}(x) &= \nabla V \cdot f(x) \\ &= \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}\right][f_1(x), f_2(x)]^T \\ &= [mgl\sin x_1, ml^2x_2][x_2, -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2]^T \\ &= -kl^2x_2^2. \end{aligned}$$

Thus $\dot{V}(x)$ *is negative* semi-definite*. It is not negative definite since* $\dot{V}(x) = 0$ *for* $x_2 = 0$*, regardless of the value of* x_1 *(thus* $\dot{V}(x) = 0$ *along the* x_1 *axis). We conclude that the origin is stable by Theorem 1, but cannot conclude asymptotic stability.* □

The result is disappointing. We know that a pendulum with friction converges to $x = 0$. This example emphasizes the fact that *all* of the Lyapunov theorems provide *sufficient* but not *necessary* conditions for stability.

Example 5 : *Consider the following system:*

$$\begin{aligned}\dot{x}_1 &= x_1(x_1^2 + x_2^2 - \beta^2) + x_2 \\ \dot{x}_2 &= -x_1 + x_2(x_1^2 + x_2^2 - \beta^2).\end{aligned}$$

To study the equilibrium point at the origin, we define $V(x) = 1/2(x_1^2 + x_2^2)$. *We have*

$$\begin{aligned}\dot{V}(x) &= \nabla V \cdot f(x) \\ &= [x_1, x_2][x_1(x_1^2 + x_2^2 - \beta^2) + x_2^2, -x_1 x_2(x_1^2 + x_2^2 - \beta^2)]^T \\ &= x_1^2(x_1^2 + x_2^2 - \beta^2) + x_2^2(x_1^2 + x_2^2 - \beta^2) \\ &= (x_1^2 + x_2^2)(x_1^2 + x_2^2 - \beta^2).\end{aligned}$$

Thus, $V(x) > 0$ *and* $\dot{V}(x) \leq 0$*, provided that* $(x_1^2 + x_2^2) < \beta^2$*, and it follows that the origin is an asymptotically stable equilibrium point.* □

5 Asymptotic Stability in the Large

Definition 7 : *The equilibrium state x_e is globally asymptotically stable (or A.S. in the large), if it is stable and every motion converges to the equilibrium as $t \to \infty$.*

Question Can we infer that if the conditions of Theorem 2 hold in the whole space R^n, then the asymptotic stability of the equilibrium is global?

Answer: No! and the next example illustrates this.

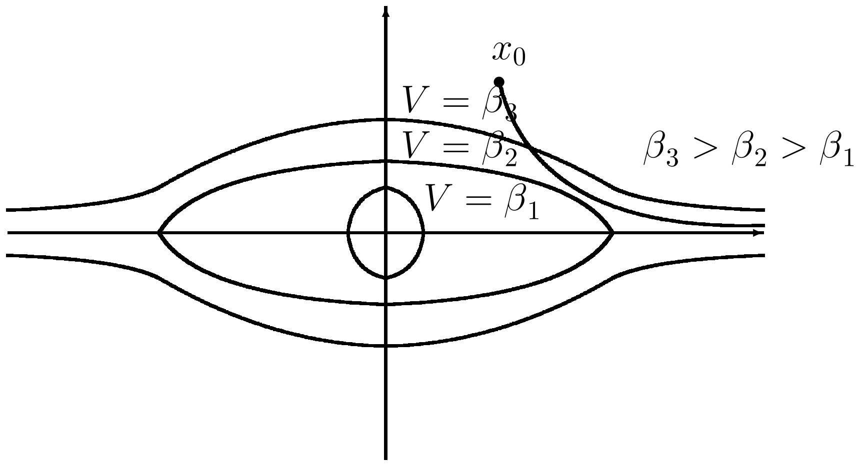

Figure 4: The curves $V(x) = \beta$.

Example 6 : *Consider the following positive definite function:*

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2.$$

The region $V(x) \leq \beta$ is closed for values of $\beta < 1$. However, if $\beta > 1$, the surface is open. Figure 4 shows that an initial state can diverge from the equilibrium state at the origin while moving towards lower energy curves. $\square$

Definition 8 : *Let $V : D \to R$ be a continuously differentiable function. Then $V(x)$ is said to be radially unbounded if*

$$V(x) \to \infty \quad as \quad \|x\| \to \infty.$$

Theorem 3 : *(Global Asymptotic Stability) Under the conditions of Theorem 2, if $V(\cdot)$ is radially unbounded then $x = 0$ is globally asymptotically stable.*

6 Positive Definite Functions Revisited

Definition 9 *: A continuous function $\alpha : [0, a) \to R^+$ is said to be in the class $\mathcal{K}$ if*

(i) $\alpha(0) = 0$.

(ii) It is strictly increasing.

α is said to be in the class $\mathcal{K}_\infty$ if in addition $\alpha : R^+ \to R^+$ and $\alpha(r) \to \infty$ as $r \to \infty$.

In the sequel, B_r represents the ball

$$B_r = \{x \in R^n : \|x\| \le r\}.$$

Lemma 4 *: $V : D \to R$ is positive definite if and only if there exists class $\mathcal{K}$ functions α_1 and α_2 such that*

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|) \quad \forall x \in B_r \subset D.$$

Moreover, if $D = R^n$ and $V(\cdot)$ is radially unbounded then α_1 and α_2 can be chosen in the class $\mathcal{K}_\infty$.

Example 7 *: Let $V(x) = x^T P x$, where P is a constant positive definite symmetric matrix. Denote $\lambda_{min}(P)$ and $\lambda_{max}(P)$ the minimum and maximum eigenvalues of P, respectively. We have:*

$$\begin{aligned} \lambda_{min}(P)\|x\|^2 &\le x^T P x \le \lambda_{max}(P)\|x\|^2 \\ \lambda_{min}(P)\|x\|^2 &\le V(x) \le \lambda_{max}(P)\|x\|^2. \end{aligned}$$

Thus, $\alpha_1, \alpha_2 : [0, \infty) \to R^+$, and are defined by

$$\begin{aligned} \alpha_1(x) &= \lambda_{min}(P)\|x\|^2 \\ \alpha_2(x) &= \lambda_{max}(P)\|x\|^2. \end{aligned}$$

□

Lemma 5 *: $x = 0$ is stable if and only if there exists a class $\mathcal{K}$ function $\alpha(\cdot)$ and a constant δ such that*

$$\|x(0)\| \;<\; \delta \quad \Rightarrow \quad \|x(t)\| \leq \alpha(\|x(0)\|) \quad \forall t \geq 0. \tag{3}$$

A stronger class of functions is needed in the definition of asymptotic stability.

Definition 10 *: A continuous function $\beta : [0, a) \times R^+ \to R^+$ is said to be in the class $\mathcal{KL}$ if*

(i) For fixed s, $\beta(r, s)$ is in the class $\mathcal{K}$ with respect to r.

(ii) For fixed r, $\beta(r, s)$ is decreasing with respect to s.

(iii) $\beta(r, s) \to 0$ as $s \to \infty$.

Lemma 6 *: The equilibrium $x = 0$ of the system (1) is asymptotically stable if and only if there exists a class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ and a constant δ such that*

$$\|x(0)\| \;<\; \delta \;\; \Rightarrow \;\; \|x(t)\| \leq \beta(\|x(0)\|, t) \quad \forall t \geq 0. \tag{4}$$

6.1 Exponential Stability

Theorem 7 : *Suppose that all the conditions of Theorem 2 are satisfied, and in addition assume that there exist positive constants* K_1, K_2, K_3 *and* p *such that*

$$\begin{array}{rcccl} K_1\|x\|^p & \le & V(x) & \le & K_2\|x\|^p \\ \dot{V}(x) & \le & -K_3\|x\|^p. & & \end{array}$$

Then the origin is exponentially stable. Moreover, if the conditions hold globally, the $x = 0$ *is globally exponentially stable.*

Proof: According to the assumptions of Theorem 7, the function $V(x)$ satisfies Lemma 4 with $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$, satisfying somewhat strong conditions. Indeed, by assumption

$$\begin{array}{rcccl} K_1\|x\|^p & \le & V(x) & \le & K_2\|x\|^p \\ \dot{V}(x) & \le & -K_3\|x\|^p & & \\ & \le & -\frac{K_3}{K_2}V(x) & & \end{array}$$

i.e.

$$\begin{array}{rcccl} \dot{V}(x) & \le & -\frac{K_3}{K_2}V(x) & & \\ \Rightarrow V(x) & \le & V(x_0)e^{-(K_3/K_2)t} & & \\ \Rightarrow \|x\| & \le & [\frac{V(x)}{K_1}]^{1/p} & \le & [\frac{V(x_0)e^{-(K_3/K_2)t}}{K_1}]^{1/p} \end{array}$$

or

$$\|x(t)\| \le \|x_0\| \ [\frac{K_2}{K_1}]^{1/p} \ e^{-(K_3/\rho K_2)t}.$$

□

7 The Invariance Principle

Asymptotic stability is always more desirable that stability. Lyapunov functions often fail to identify asymptotic stability. We now study an improvement over the Lyapunov theorems studied earlier.

Definition 11 : *A set M is said to be an invariant set with respect to the dynamical system $\dot{x} = f(x)$ if:*

$$x(0) \in M \quad \Rightarrow \quad x(t) \in M \quad \forall t \in R^+.$$

Example 8 : *Any equilibrium point is an invariant set, since if at $t = 0$ we have $x(0) = x_e$, then $x(t) = x_e \quad \forall t \geq 0$.* □

Example 9 : *For autonomous systems, any trajectory is an invariant set.* □

Example 10 : *A limit cycle is an invariant set (a special case of Example 9).* □

Example 11 : *If $V(x)$ is continuously differentiable (not necessarily positive definite) and satisfies $\dot{V}(x) < 0$ along the solutions of $\dot{x} = f(x)$, then the set Ω_l defined by*

$$\Omega_l = \{x \in R^n : V(x) \leq l\}$$

is an invariant set. □

Theorem 8 : *The equilibrium point $x = 0$ of the autonomous system (1) is asymptotically stable if there exists a function $V(x)$ satisfying*

(i) $V(x)$ *positive definite* $\forall x \in D$, *where we assume that* $0 \in D$.

(ii) $\dot{V}(x)$ *is negative semi definite in a bounded region* $R \subset D$.

(iii) $\dot{V}(x)$ *does not vanish identically along any trajectory in* R, *other than the null solution* $x = 0$.

Example 12 : *Consider again the pendulum with friction of Example 4:*

$$\dot{x}_1 = x_2 \tag{5}$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2. \tag{6}$$

Again

$$\begin{aligned} V(x) &> 0 \quad \forall x \in (-\pi, \pi) \times R, \\ \dot{V}(x) &= -kl^2 x_2^2 \end{aligned} \tag{7}$$

which is negative semi definite *since* $\dot{V}(x) = 0$ *for all* $x = [x_1, 0]^T$ *(so* $x = 0$ *stable but cannot conclude AS). We now apply Theorem 8. Conditions (i) and (ii) of Theorem 8 are satisfied in the region*

$$R = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with $-\pi < x_1 < \pi$, *and* $-a < x_2 < a$, *for any* $a \in R^+$. *We now check condition (iii), that is, we check whether* $\dot{V}$ *can vanish identically along the trajectories trapped in* R, *other than the null solution.*

By (7) we have

$$\dot{V}(x) = 0 \;\;\Rightarrow\;\; 0 = -kl^2 x_2^2 \quad \Longleftrightarrow x_2 = 0$$

$$\textit{thus} \quad x_2 = 0 \quad \forall t \;\;\Rightarrow\;\; \dot{x}_2 = 0$$

and by (6), we obtain

$$0 = \frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \quad \textit{andthus } x_2 = 0 \Rightarrow \sin x_1 = 0$$

restricting x_1 *to* $x_1 \in (-\pi, \pi)$ *we have that the last condition is satisfied if and only if* $x_1 = 0$. *Thus,* $\dot{V}(x) = 0$ *does not vanish identically along any trajectory other than* $x = 0$. *Thus* $x = 0$ *is asymptotically stable by Theorem 8.* □

Theorem 9 *: The null solution $x = 0$ of the autonomous system (1) is asymptotically stable in the large if the assumptions of theorem 8 hold in the entire state space and $V(\cdot)$ is radially unbounded.*

Example 13 *:*

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_2 - \alpha x_1 - (x_1 + x_2)^2 x_2.
\end{aligned}$$

where α is a positive scalar.
To study the stability of $x = 0$ we define $V(x) = \alpha x_1^2 + x_2^2$ (Radially unbounded). Thus differentiating $V(x)$

$$\begin{aligned}
\dot{V}(x) &= \frac{\partial V}{\partial x} f(x) \\
&= -2x_2^2[1 + (x_1 + x_2)^2]
\end{aligned}$$

and $V(x) > 0$ and $\dot{V}(x) \leq 0$ since $\dot{V}(x) = 0$ for $x = (x_1, 0)$. Assume now that $\dot{V} = 0$:

$$\dot{V} = 0 \quad \Longleftrightarrow \quad x_2 = 0\ , \quad x_2 = 0 \quad \forall t \quad \Rightarrow \quad \dot{x}_2 = 0$$

$$\dot{x}_2 = 0 \quad \Rightarrow \quad -x_2 - \alpha x_1 - (x_1 + x_2)^2 x_2 = 0$$

and considering the fact that $x_2 = 0$, the last equation implies that $x_1 = 0$. It follows that $\dot{V}(x)$ does not vanish identically along any solution other than $x = [0, 0]^T$. Thus, $x = 0$ is globally asymptotically stable. □

Theorem 10 *: (LaSalle's theorem) Let $V : D \to R$ be a continuously differentiable function and assume that*

(i) *$M \subset D$ is a compact set, invariant with respect to the solutions of (1).*

(ii) *$\dot{V} \leq 0$ in M.*

(iii) *$E : \{x : x \in M, \text{and } \dot{V} = 0\}$; that is, E is the set of all points of M such that $\dot{V} = 0$.*

(iv) *N: is the largest invariant set in E.*

Then every solution starting in M approaches N as $t \to \infty$.

8 Region of Attraction

Example 14 *: Consider the system defined by*

$$\begin{cases} \dot{x}_1 &= 3x_2 \\ \dot{x}_2 &= -5x_1 + x_1^3 - 2x_2. \end{cases}$$

We are interested in the stability of $x = 0$*. Consider*

$$\begin{aligned} V(x) &= 12x_1^2 - x_1^4 + 6x_1x_2 + 6x_2^2 \\ &= 3(x_1 + 2x_2)^2 + 9x_1^2 + 3x_2^2 - x_1^4 \qquad (8) \\ \dot{V}(x) &= -6x_2^2 - 30x_1^2 + 6x_1^4. \qquad (9) \end{aligned}$$

According to Theorem 2, if $V(x) > 0$ *and* $\dot{V} < 0$ *in* $D - \{0\}$*, then* $x = 0$ *is "locally" asymptotically stable.*

Studying V *and* $\dot{V}$ *we conclude that defining* D *by*

$$D = \{x \in R^2 : -1.6 < x_1 < 1.6\} \qquad (10)$$

we have that $V(x) > 0$ *and* $\dot{V} < 0, \forall x \in D - \{0\}$*.*
Question: Can we conclude that any solution starting in D *converges to the origin? No!*

Plotting the trajectories as shown in we see that, for example, the trajectory initiating at the point $x_1 = 0, x_2 = 4$ *is quickly divergent from* $x = 0$ *even though the point* $(0, 4) \in D$*.*

The problem is this: D *is not an invariant set and there are no guarantees that trajectories starting in* D *will remain within* D*. Thus, once a trajectory crosses the border* $|x_1| = \sqrt{5}$ *there are no guarantees that* $\dot{V}(x)$ *will be negative.* □

We now study how to estimate the region of attraction.

Definition 12 *: Let* $\psi(x, t)$ *be the trajectories of the systems (1) with initial condition* x *at* $t = 0$*. The region of attraction to the equilibrium point* x_e*, denoted* R_A*, is defined by*

$$R_A = \{x \in D : \psi(x, t) \to x_e, \ as \ \ t \to \infty\}.$$

We nos estimate this region based on LaSalle's Theorem.

Theorem 11 *: Let $V : D \to R$ be a continuous differentiable function and assume that x_e is an equilibrium point and*

(i) *$M \subset D$ is a compact set containing x_e, invariant with respect to the solutions of (1).*

(ii) *$\dot{V}$ is such that*

$$\begin{aligned} \dot{V} &< 0 \;\; \forall x \neq x_e \in M. \\ \dot{V} &= 0 \;\;\; if\; x = x_e. \end{aligned}$$

Under these conditions we have that

$$M \subset R_A.$$

***Example 15** :*

$$\begin{aligned}
\dot{x} &= -kx + x^3 \\
V &= \frac{1}{2}x^2 \\
\\
\dot{V} &= x\dot{x} \\
&= -kx^2 + x^4 \\
&= -\left(K - x^2\right)x^2 \\
\dot{V} &\le -\beta x^2 \qquad \mathit{for}\; k > x^2 \\
\dot{V} &\le -\beta x^2 \qquad \mathit{for}\; k > 2V(t) \qquad \{V(0) > V(t)\} \\
\dot{V} &\le -\beta x^2 \qquad \mathit{for}\; k > 2V(0) \\
\dot{V} &\le -\beta x^2 \qquad \mathit{for}\; k > x^2(0) \\
\dot{V} &\le -2\beta V(t) \qquad \mathit{for}\; k > x^2(0) \\
V(t) &\le V(0)\exp(-2\beta t) \qquad \mathit{for}\; K > x^2(t) \\
\frac{1}{2}x^2(t) &\le \frac{1}{2}x^2(0)\exp(-2\beta t) \\
|x(t)| &\le |x(0)|\exp(-\beta t) \qquad \mathit{for}\; K > |x(0)|^2
\end{aligned}$$

9 Analysis of Linear Time-Invariant Systems

Consider the autonomous linear time-invariant system given by

$$\dot{x} = Ax, \qquad A \in R^{n\times n} \tag{11}$$

and let $V(\cdot)$ be defined as follows

$$V(x) = x^T P x \tag{12}$$

where $P \in R^{n\times n}$ is (i) symmetric and (ii) positive definite (iii) P is a constant. Thus $V(\cdot)$ is positive definite. Also

$$\begin{aligned}\dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T(A^T P + PA)x\end{aligned}$$

or

$$\dot{V} = -x^T Q x \tag{13}$$

$$PA + A^T P = -Q. \tag{14}$$

Here the matrix Q is symmetric, since

$$Q^T = -(PA + A^T P)^T = -(A^T P + AP) = Q$$

If Q is positive definite, then $\dot{V}(\cdot)$ is negative definite and the origin is (globally) asymptotically stable. To analyze the positive definiteness of the pair of matrices (P, Q) we need two steps:

(i) Choose an arbitrary symmetric, positive definite matrix Q.

(ii) Find P that satisfies equation (14) and verify that it is positive definite.

Equation (14) appears very frequently in the literature and is called *Algebraic Lyapunov equation*.

The procedure described above for the stability analysis based on the pair (P, Q) depends on the existence of a unique solution of the Lyapunov equation for a given matrix A. The following theorem guarantees the existence of such a solution.

Theorem 12 : *The eigenvalues λ_i of a matrix $A \in R^{n\times n}$ satisfy $\Re e(\lambda_i) < 0$ if and only if for* any *given symmetric positive definite matrix Q there exists a unique positive definite symmetric matrix P satisfying the Lyapunov equation (14)*

Proof: Assume first that given $Q > 0$, $\exists\ P > 0$ satisfying (14). Thus $V = x^T P x > 0$ and $\dot{V} = -x^T Q x < 0$ and asymptotic stability follows from Theorem 2.

For the converse assume that $\Re e(\lambda_i) < 0$ and given Q, define P as follows:

$$P \;=\; \int_0^\infty e^{A^T t} Q e^{At}\, dt$$

which is symmetric. We claim that it is also positive definite. To see this, assume the opposite: i.e. that $\exists x \neq 0$ such that $x^T P x = 0$. But then

$$\begin{aligned}
x^T P x = 0 \quad &\Rightarrow \quad \int_0^\infty x^T e^{A^T t} Q e^{At} x\, dt \;=\; 0 \\
&\Rightarrow \quad \int_0^\infty y^T Q y\, dt \;=\; 0 \quad \textit{with } y = e^{At}x \\
&\Longleftrightarrow \quad y = e^{At}x = 0 \quad \forall t \geq 0 \\
&\Longleftrightarrow \quad x = 0
\end{aligned}$$

since e^{At} is nonsingular $\forall t$. This contradicts the assumption. Thus $P > 0$. We now show that P satisfies the Lyapunov equation

$$\begin{aligned}
PA + A^T P \;&=\; \int_0^\infty e^{A^T t} Q e^{At} A\, dt \;+\; \int_0^\infty A^T e^{A^T t} Q e^{At}\, dt \\
&=\; \int_0^\infty \frac{d}{dt}(e^{A^T t} Q e^{At})\, dt \\
&=\; e^{A^T t} Q e^{At}\, |_0^\infty \;=\; -Q
\end{aligned}$$

To complete the proof, there remains to show that this P is unique. Suppose that there is another solution $\widetilde{P} \neq P$. Then

$$\begin{aligned}
(P - \widetilde{P})A + A^T(P - \widetilde{P}) \;&=\; 0 \\
\Rightarrow e^{A^T t}\ \left[(P - \widetilde{P})A + A^T(P - \widetilde{P})\right] e^{At} \;&=\; 0 \\
\Rightarrow \frac{d}{dt}\ \left[e^{A^T t}(P - \widetilde{P})e^{At}\right] \;&=\; 0
\end{aligned}$$

which implies that $e^{A^T t}(P - \widetilde{P})e^{At}$ is constant $\forall t$. This can be the case if and only if $P - \widetilde{P} = 0$, or equivalently, $P = \widetilde{P}$. □

10 Instability

Theorem 13 *: (Chetaev) Consider the autonomous dynamical systems (1) and assume that $x = 0$ is an equilibrium point. Let $V : D \to R$ have the following properties:*

(i) $V(0) = 0$

(ii) $\exists\ x_0 \in R^n$, *arbitrarily close to* $x = 0$, *such that* $V(x_0) > 0$

(iii) $\dot{V} > 0\ \forall\ x \in U$, *where the set* U *is defined as follows:*

$$U = \{x \in D : \|x\| \leq \epsilon, \textit{and}\ V(x) > 0\}.$$

Under these conditions, $x = 0$ *is unstable.*

Example 16 *: Consider again the system of Example 3.20 (textbook)*

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1(\beta^2 - x_1^2 - x_2^2) \\ \dot{x}_2 &= -x_1 + x_2(\beta^2 - x_1^2 - x_2^2) \end{aligned}$$

The origin of this system is an unstable equilibrium point. We now verify this result using Chetaev's result. Let $V(x) = 1/2(x_1^2 + x_2^2)$. *Thus we have that* $V(0) = 0$, *and moreover* $V(x) > 0\ \forall\ x \in R^2 \neq 0$, *i.e.,* $V(\cdot)$ *is positive definite. Also*

$$\begin{aligned} \dot{V} &= (x_1, x_2) f(x) \\ &= (x_1^2 + x_2^2)(\beta^2 - x_1^2 - x_2^2). \end{aligned}$$

Defining the set U *by*

$$U = \{x \in R^2 : \|x\| \leq \epsilon,\ \ 0 < \epsilon < \beta\}$$

we have that $V(x) > 0\ \forall\ x \in U, x \neq 0$, *and* $\dot{V} > 0\ \forall\ x \in U,\ x \neq 0$. *Thus the origin is unstable, by Chetaev's result.*

$\square$