Chapter 4: Lyapunov Stability II: Nonautonomous Systems

1 Definitions

Consider the non-autonomous system:

$$\dot{x} = f(x,t) \qquad\qquad f : D \times R^+ \to R^n \tag{1}$$

where f is locally Lipschitz in x and piecewise continuous in t on $D \times [0, \infty)$. $x = 0 \in D$ is an equilibrium point of (1) if

$$f(0,t) = 0 \qquad \forall t \geq t_0.$$

Definition 1 *: $x = 0$ is said to be*

- Stable *at t_0 if given $\epsilon > 0$, $\exists \delta = \delta(\epsilon, t_0) > 0$:*

$$\|x(0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < \epsilon \quad \forall t \geq t_0 > 0 \tag{2}$$

- Convergent *at t_0 if there exists $\delta_1 = \delta_1(t_0) > 0$:*

$$\|x(0)\| < \delta_1 \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = 0. \tag{3}$$

  Equivalently (and more precisely), x_0 is convergent at t_0 if for any given $\epsilon_1 > 0, \exists T = T(\epsilon_1, t_0)$ such that

$$\|x(0)\| < \delta_1 \quad \Rightarrow \quad \|x(t)\| < \epsilon_1 \quad \forall t \geq t_0 + T \tag{4}$$

- Asymptotically stable *at t_0 if it is both stable and convergent.*
- Unstable *if it is not stable.*

Notice the inclusion of the initial time t_0. This dependence is not desirable and motivates the several notions of uniform stability.

Definition 2 *: The equilibrium point $x = 0$ of the system (1) is said to be*

- Uniformly stable *if any given* $\epsilon > 0,\ \exists \delta = \delta(\epsilon) > 0:$

$$\|x(0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < \epsilon \quad \forall t \geq t_0 > 0 \tag{5}$$

- Uniformly convergent *if there is* $\delta_1 > 0$, *independent of* t_0, *such that*

$$\|x_0\| < \delta_1 \quad \Rightarrow \quad x(t) \to 0 \ \ as \ \ t \to \infty.$$

  Equivalently, $x = 0$ *is uniformly convergent if for any given* $\epsilon_1 > 0, \exists T = T(\epsilon_1)$ *such that*

$$\|x(0)\| < \delta_1 \quad \Rightarrow \quad \|x(t)\| < \epsilon_1 \quad \forall t \geq t_0 + T$$

- Uniformly asymptotically stable *if it is uniformly stable and uniformly convergent.*
- Globally uniformly asymptotically stable *if it is uniformly asymptotically stable and every motion converges to the origin.*

2 Positive Definite Functions

In the following definitions we consider a function $W : D \times R^+ \to R$. Furthermore we assume that

(i) $0 \in D$.

(ii) $W(x, t)$ is continuous and has continuous partial derivatives with respect to all of its arguments.

Definition 3 *: $W(\cdot, \cdot)$ is said to be positive semi definite in D if*

(i) $W(0, t) = 0 \qquad \forall t \in R^+$

(ii) $W(x, t) \geq 0 \qquad \forall x \neq 0, x \in D$

Definition 4 : *$W(\cdot,\cdot)$ is said to be positive definite in D if*

(i) $W(0,t) = 0 \qquad \forall t \in R^+$

(ii) $\exists$ *a time-invariant positive definite function $V_1(x)$ such that*

$$V_1(x) \ \leq \ W(x,t) \qquad \forall x \in D. \tag{6}$$

Definition 5 : *$W(\cdot,\cdot)$ is said to be decrescent in D if there exists a positive definite function $V_2(x)$ such that*

$$|W(x,t)| \ \leq V_2(x) \qquad \forall x \in D. \tag{7}$$

($\Rightarrow$ every time-invariant positive definite function is decrescent.)

Definition 6 : *$W(\cdot,\cdot)$ is radially unbounded if*

$$W(x,t) \to \infty \qquad as \quad \|x\| \to \infty$$

uniformly on t. Equivalently, $W(\cdot,\cdot)$ is radially unbounded if given $M, \exists N > 0$ such that

$$W(x,t) > M$$

for all t, provided that $\|x\| > N$.

Remarks: Consider now function $W(x,t)$. By Definition 6, $W(\cdot,\cdot)$ is positive definite in D if and only if $\exists V_1(x)$ such that

$$V_1(x) \ \leq \ W(x,t) \ , \quad \forall x \in D \tag{8}$$

this implies the existence of a class κ function $\alpha_1(\cdot)$ such that

$$\alpha_1(\|x\|) \ \leq V_1(x) \ \leq \ W(x,t) \ , \quad \forall x \in B_r \subset D. \tag{9}$$

If in addition $W(\cdot,\cdot)$ is decrescent, then, according to Definition (7) there exists V_2:

$$W(x,t) \leq V_2(x) \ , \quad \forall x \in D \tag{10}$$

this implies the existence of a class κ function $\alpha_2(\cdot)$ such that

$$W(x,t) \leq V_2(x) \leq \alpha_2(\|x\|) \ , \quad \forall x \in B_r \subset D. \tag{11}$$

It follows that $W(\cdot,\cdot)$ is positive definite and decrescent if and only if there exist positive definite functions $V_1(\cdot)$ and $V_2(\cdot)$, such that

$$V_1(x) \ \leq \ W(x,t) \ \leq \ V_2(x) \ , \quad \forall x \in D \tag{12}$$

which in turn implies the existence of $\alpha_1(\cdot)$ and $\alpha_2(\cdot) \in \mathcal{K}$ such that

$$\alpha_1(\|x\|) \ \leq \ W(x,t) \ \leq \ \alpha_2(\|x\|) \ , \quad \forall x \in B_r \subset D. \tag{13}$$

Finally, $W(\cdot,\cdot)$ is positive definite, decrescent and radially unbounded if and only if $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ can be chosen in the class $\mathcal{K}_\infty$.

2.1 Examples

Let $x = [x_1, x_2]^T$ and study several functions $W(x,t)$.

Example 1 : *Let* $W_1(x,t) = (x_1^2 + x_2^2)e^{-\alpha t} \quad \alpha > 0$. *We have:*

(i) $W_1(0,t) = 0\ e^{-\alpha t} = 0$.

(ii) $W_1(x,t) > 0 \quad \forall x \neq 0, \quad \forall t \in R$.

However, $\lim_{t\to\infty} W_1(x,t) = 0 \quad \forall x$. *Thus,* $W_1(\cdot,\cdot)$ *is positive semi definite, but* not *positive definite.* $\square$

Example 2 : *Let*

$$\begin{aligned} W_2(x,t) &= \frac{(x_1^2 + x_2^2)(t^2+1)}{(x_1^2+2)} \\ &= V_2(x)(t^2+1), \qquad V_2(x) \triangleq \frac{(x_1^2 + x_2^2)}{(x_1^2+2)}. \end{aligned}$$

Thus, $W_2(x,t) \geq V_2(x) > 0\ \forall x \in R^2$ *and* $W_2(\cdot,\cdot)$ *positive definite. Also*

$$\lim_{t\to\infty} W_2(x,t) = \infty \quad \forall x \in R^2.$$

Thus we cannot find a positive definite function $V(\cdot)$ *such that* $|W_2(x,t)| \leq V(x)\ \forall x$. *Thus* $W_2(x,t)$ *is not decrescent.* $|W_2(x,t)|$ *is not radially unbounded since* $W_2(x,t) \to \infty$ *as* $x_1 \to \infty$. $\square$

Example 3 : *Let*

$$W_3(x,t) = (x_1^2 + x_2^2)(t^2+1)$$

$W_3(x)$ *is positive definite, radially unbounded and not decrescent.* $\square$

Example 4 : *Let*

$$W_4(x,t) = \frac{(x_1^2 + x_2^2)}{(x_1^2+1)}.$$

Thus, $W_4(\cdot,\cdot) > 0 \forall x \in R^2$ *and is positive definite. It is not time-dependent, and so it is decrescent. It is not radially unbounded.* $\square$

Example 5 *Let*

$$\begin{aligned} W_5(x,t) &= \frac{(x_1^2 + x_2^2)(t^2+1)}{(t^2+2)} \\ &= V_5(x)\frac{(t^2+1)}{(t^2+2)}, \qquad V_5(x) \triangleq (x_1^2 + x_2^2). \end{aligned}$$

Thus, $W_5(x,t) \geq k_1 V_5(x)$ for some constant k_1, which implies that $W_5(\cdot,\cdot)$ is positive definite. It is decrescent since

$$|W_5(x,t)| \leq k_2 V_5(x) \; \forall x \in$$

It is also radially unbounded since

$$W_5(x,t) \to \infty \qquad as \;\; \|x\| \to \infty.$$

3 Stability Theorems

Consider the system (1) and assume that $x = 0$ is an equilibrium state of:

$$f(0,t) = 0 \qquad \forall t \in .$$

In the following theorems, we assume that $W(\cdot,\cdot)$ has continuous partial derivatives in all of its arguments.

Theorem 1 *: (Stability Theorem) If in a neighborhood D of $x = 0$ there exists $W(\cdot,\cdot) : D \times [0,\infty) \to R$ such that*

(i) $W(x,t)$ is positive definite.

(ii) The derivative of $W(\cdot,\cdot)$ along any solution of (1) is negative semi definite in D, then

the equilibrium state is stable. If $W(x,t)$ is also decrescent then the origin is uniformly stable.

Theorem 2 *: (Uniform Asymptotic Stability) If in a neighborhood D of the equilibrium state $x = 0$ there exists $W(\cdot,\cdot) : D \times [0,\infty) \to R$ such that*

(i) $W(x,t)$ is (a) positive definite, and (b) decrescent, and

(ii) The derivative of $\dot{W}(x,t)$ is negative definite in D, then

the equilibrium state is uniformly asymptotically stable.

The assumptions in Theorem 2 mean that:

(i) $V_1(x) \leq W(x,t) \leq V_2(x) \quad \forall x \in D, \forall t$

(ii) $\frac{\partial W}{\partial t} + \nabla W f(x,t) \leq -V_3(x) \quad \forall x \in D, \forall t$

where $V_i, i = 1,2,3$ are positive definite functions in D.

Theorem 3 : *(Global Uniform Asymptotic Stability) If there exists $W(\cdot,\cdot) : R^n \times [0,\infty) \rightarrow R$ such that*

(i) *$W(x,t)$ is (a) positive definite, and (b) decrescent, and radially unbounded $\forall x \in R^n$, and such that*

(ii) *The derivative of $\dot{W}(x,t)$ is negative definite $\forall x \in R^n$, then*

the equilibrium state at $x = 0$ is globally uniformly asymptotically stable.

Theorem 4 : *Suppose that all the conditions of Theorem 2 are satisfied, and in addition assume that there exist positive constants K_1, K_2, and K_3 such that*

$$\begin{array}{rcl} K_1\|x\|^p & \leq & W(x,t) \quad \leq \; K_2\|x\|^p \\ \dot{W}(x) & \leq & -K_3\|x\|^p. \end{array}$$

Then the origin is exponentially stable. Moreover, if the conditions hold globally, the $x = 0$ is globally exponentially stable.

Example 6 : *Consider the following system:*

$$\begin{cases} \dot{x}_1 &= -x_1 - e^{-2t}x_2 &= f_1 \\ \dot{x}_2 &= x_1 - x_2. &= f_2 \end{cases}$$

To study the stability of $x = 0$, let

$$W(x,t) = x_1^2 + (1 + e^{-2t})x_2^2.$$

Clearly

$$V_1(x) \;=\; (x_1^2 + x_2^2) \;\leq\; W(x,t) \;\leq\; (x_1^2 + 2x_2^2) \;=\; V_2(x)$$

thus, we have that

- *$W(x,t)$ is positive definite, since $V_1(x) \leq W(x,t)$, with V_1 positive definite in R^2.*
- *$W(x,t)$ is decrescent, since $W(x,t) \geq V_2(x)$, with V_2 also positive definite in R^2.*

Then

$$\begin{aligned}
\dot{W}(x,t) &= \frac{\partial W}{\partial x} f(x,t) + \frac{\partial W}{\partial t} \\
&= -2[x_1^2 - x_1x_2 + x_2^2(1 + 2e^{-2t})] \\
\dot{W}(x,t) &\leq -2[x_1^2 - x_1x_2 + x_2^2]. \\
\dot{W}(x,t) &\leq -[x_1^2 + x_2^2] - [x_1^2 - x_1x_2 + x_2^2] \\
\dot{W}(x,t) &\leq -[x_1^2 + x_2^2] - (x_1 - x_2)^2 \\
\dot{W}(x,t) &\leq -x_1^2 - x_2^2
\end{aligned}$$

It follows that $\dot{W}(x,t)$ is negative definite and the origin is globally asymptotically stable.

□

4 Perturbation Analysis

In practice, a model can "approximate" a true system. The difference is referred to as *uncertainty*. Consider a dynamical system of the form

$$\dot{x} = f(x,t) + g(x,t) \tag{14}$$

$g(x,t)$: perturbation term used to represent uncertainty .

<u>Question</u>: Suppose that $\dot{x} = f(x,t)$ has an asymptotically stable equilibrium point; what can be said about $\dot{x} = f(x,t) + g(x,t)$?

Theorem 5 : *Let $x = 0$ be an equilibrium point of the system (14) and assume that there exist $W(\cdot,\cdot) : D \times [0,\infty) \to R$ such that*

(i) $k_1\|x\|^2 \le W(x,t) \le k_2\|x\|^2$.

(ii) $\frac{\partial W}{\partial t} + \nabla W f(x,t) \le -k_3\|x\|^2$.

(iii) $\|\nabla W\| \le k_4\|x\|$.

Then if the perturbation $g(x,t)$ satisfies the bound

(iv) $\|g(x,t)\| \le k_5\|x\|, \qquad (k_3 - k_4 k_5) > 0$

the origin is exponentially stable (globally if the assumptions hold globally).

Proof: By (i) $W(x,t)$ is positive definite and decrescent, with $\alpha_1(\|x\|) = k_1\|x\|^2$ and $\alpha_2(\|x\|) = k_2\|x\|^2$. Moreover, $W(\cdot,\cdot)$ is radially unbounded. Assumption (ii) implies that, $\dot{W}(x,t)$ is negative definite along the trajectories of the system $\dot{x} = f(x,t)$ (i.e., ignoring $g(x,t)$. Thus, (i) and (ii) imply that $x = 0$ is uniformly asymptotically stable *for the nominal system* $\dot{x} = f(x,t)$.

We now find $\dot{W}(\cdot,\cdot)$ along the trajectories of the perturbed system (14). We have

$$\begin{aligned} \dot{W} &= \underbrace{\frac{\partial W}{\partial t} + \nabla W f(x,t)}_{\leq -k_3\|x\|^2} + \underbrace{\nabla W g(x,t)}_{\leq k_4 k_5 \|x\|^2} \\ \Rightarrow \quad \dot{W} &\leq -(k_3 - k_4 k_5)\|x\|^2 \quad < 0 \end{aligned}$$

since $(k_3 - k_4 k_5) > 0$ by assumption. The result then follows by Theorems 2–3 (along with Theorem 4). $\square$

5 Discrete-Time Systems

Consider a *discrete-time systems* of the form

$$x(k+1) = f(x(k), k) \tag{15}$$

where $k \in Z^+, x(k) \in R^n$, and $f : R^n \times Z \rightarrow R^n$.

6 Discretization

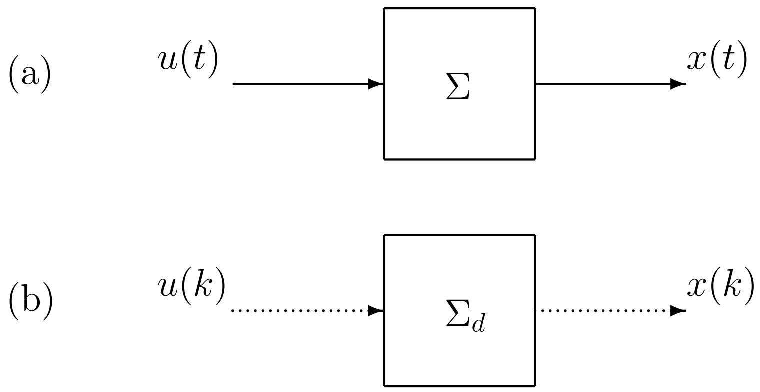

Figure 1: (a) Continuous-time system Σ; (b) discrete-time system Σ_d.

Discrete-time systems $\Sigma_d : u(k) \rightarrow x(k)$ may originate by "sampling" a continuous-time system $\Sigma : u \rightarrow x$. We use the scheme shown in the Figure:

- S:*Sampler*. Reads $x(t)$ every T seconds; $x(k) = x(kT)$.
- Σ: the plant. Given u, $\Sigma : u \rightarrow x$ determines $x(t)$ by solving

$$\dot{x} = f(x, u).$$

- H:*Hold* device. Converts $u(k)$ into $u(t)$ (continuous-time)

$$u(t) = u(k) \qquad for\ kT \leq t < (k+1)T.$$

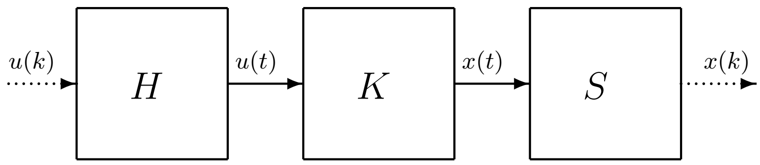

Figure 2: Discrete-time system Σ_d.

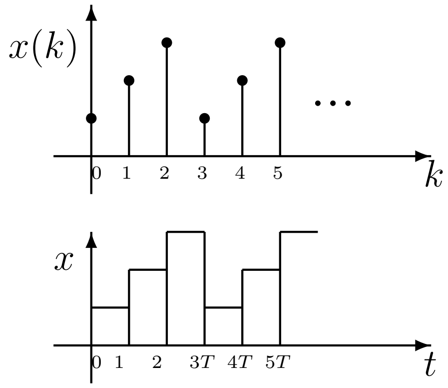

Figure 3: Action of the hold device H.

Finding Σ_d is usually impossible. There are several methods to construct approximate models. The simplest is the so-called Euler approximation. If T is small, then

$$\dot{x} = \frac{dx}{dt} = \lim_{\Delta T \to 0} \frac{x(t + \Delta T) - x(t)}{\Delta T} \approx \frac{x(t+T) - x(t)}{T}.$$

Thus

$$\dot{x} = f(x, u)$$

can be approximated by

$$x(kT + T) \approx x(kT) + T f[x(kT), u(kT)].$$

7 Stability of Discrete-Time Systems

Consider a *discrete-time systems* of the form (15)

$$x(k+1) = f(x(k), k)$$

where $k \in Z^+, x(k) \in R^n$, and $f : R^n \times Z \to R^n$, and consider the stability of an equilibrium point x_e.

7.1 Definitions

We now restate stability definitions for discrete-time systems.

Definition 7 *: The equilibrium point $x = 0$ of the system (15) is*

- Stable *at k_0 if given $\epsilon > 0$, $\exists \delta = \delta(\epsilon, k_0) > 0$:*

$$\|x_0\| < \delta \quad \Rightarrow \quad \|x(k)\| < \epsilon \quad \forall k \geq k_0 > 0. \tag{16}$$

- Uniformly stable *at k_0 if given any given $\epsilon > 0$, $\exists \delta = \delta(\epsilon) > 0$:*

$$\|x_0\| < \delta \quad \Rightarrow \quad \|x(k)\| < \epsilon \quad \forall k \geq k_0 > 0. \tag{17}$$

- Convergent *at k_0 if there exists $\delta_1 = \delta_1(k_0) > 0$:*

$$\|x_0\| < \delta_1 \quad \Rightarrow \quad \lim_{k\to\infty} x(k) = 0. \tag{18}$$

- Uniformly convergent *if for any given $\epsilon_1 > 0, \exists M = M(\epsilon_1)$ such that*

$$\|x_0\| < \delta_1 \quad \Rightarrow \quad \|x(k)\| < \epsilon_1 \quad \forall k \geq k_0 + M.$$

- Asymptotically stable *at k_0 if it is both stable and convergent.*
- Uniformly asymptotically stable *if it is both stable and uniformly convergent.*
- Unstable *if it is not stable.*

7.2 Discrete-Time Positive Definite Functions

Definition 8 *: A function $W : R^n \times Z^+ \to R$ is said to be*

- *Positive semidefinite in $D \subset R^n$ if*

  (i) $W(0, k) = 0 \quad \forall k \geq 0.$

(ii) $W(x, k) \geq 0 \quad \forall x \neq 0,\ x \in D$.

- *Positive definite in $D \subset R^n$ if*
  (i) $W(0, k) = 0 \quad \forall k \geq 0$, *and*
  (ii) $\exists$ *a time invariant positive definite function $V_1(x)$ such that*
  $$V_1(x) \leq W(x, k) \quad \forall x \in D\ \ \forall k.$$
- *$W(\cdot, \cdot)$ is said to be decrescent in $D \subset R^n$ if there exists a time-invariant positive definite function $V_2(x)$ such that*
  $$W(x, k) \leq V_2(x) \quad \forall x \in D,\ \ \forall k.$$
- *$W(\cdot, \cdot)$ is said to be radially unbounded if $W(x, k) \to \infty$ as $\|x\| \to \infty$, uniformly on k. This means that given $M > 0$, there exists $N > 0$ such that*
  $$W(x, k) > M$$
  provided that $\|x\| > N$.

7.3 Stability Theorems

Definition 9 : *The rate of change, $\Delta W(x,k)$, of the function $W(x,k)$ along the solutions of the difference equations of the system (15) is defined by*

$$\Delta W(x,k) = W(x(k+1), k+1) - W(x,k).$$

Theorem 6 : *(Lyapunov Stability Theorem for Discrete-Time Systems). If in a neighborhood D of the equilibrium state $x = 0$ of the system (15) there exists a function $W(\cdot,\cdot) : D \times Z^+ \to R$ such that*

(i) *$W(x,k)$ is positive definite.*

(ii) *The rate of change $\Delta W(x,k)$ along any solution of (15) is negative semidefinite in D, then*

the equilibrium state is stable. Moreover, if $W(x,k)$ is also decrescent, then the origin is uniformly stable.

Theorem 7 : *(Lyapunov Uniform Asymptotic Stability for Discrete-Time Systems). If in a neighborhood D of the equilibrium state $x = 0$ there exists a function $W(\cdot,\cdot) : D \times Z^+ \to R$ such that*

(i) *$W(x,k)$ is (a) positive definite, and (b) decrescent.*

(ii) *The rate of change, $\Delta W(x,k)$ is negative definite in D, then*

the equilibrium state is uniformly asymptotically stable.

Example 7 : *Consider the following discrete-time system:*

$$\begin{aligned} x_1(k+1) &= x_1(k) + x_2(k) && (19)\\ x_2(k+1) &= a x_1^3(k) + \frac{1}{2} x_2(k). && (20) \end{aligned}$$

To study the stability of the origin, we consider the (time-independent) Lyapunov function candidate $V(x) = \frac{1}{2}x_1^2(k) + 2x_1(k)x_2(k) + 4x_2^2(k)$*, which can be easily seen to be positive definite. We need to find* $\Delta V(x) = V(x(k+1)) - V(x(k))$*, we have*

$$\begin{aligned} V(x(k+1)) &= \frac{1}{2}x_1^2(k+1) + 2x_1(k+1)x_2(k+1) + 4x_2^2(k+1)\\ &= \frac{1}{2}[x_1(k) + x_2(k)]^2 + 2[x_1(k) + x_2(k)][a x_1^3(k) + \frac{1}{2}x_2(k)]\\ &\quad + 4[a x_1^3(k) + \frac{1}{2}x_2(k)]^2 \end{aligned}$$

$$V(x(k)) = \frac{1}{2}x_2^2 + 2x_1x_2 + 4x_2^2.$$

From here, after some trivial manipulations, we conclude that

$$\Delta V(x) = V(x(k+1)) - V(x(k)) = -\frac{3}{2}x_2^2 + 2a x_1^4 + 6a x_1^3 x_2 + 4a^2 x_1^6.$$

Therefore we have the following cases of interest:

- $a < 0$. *In this case,* $\Delta V(x)$ *is negative definite in a neighborhood of the origin, and the origin is locally asymptotically stable (uniformly, since the system is autonomous).*
- $a = 0$. *In this case* $\Delta V(x) = V(x(k+1)) - V(x(k)) = -\frac{3}{2}x_2^2 \leq 0$*, and thus the origin is stable.*

□