Chapter 5: Feedback Systems

Consider

$$\dot{x} = f(x, u), \qquad f(0, 0) = 0, \tag{1}$$

assume that u is obtained using state feedback:

$$u = \phi(x). \tag{2}$$

The stability of the origin can be studied substituting (2) into (1):

$$\dot{x} = f(x, \phi(x)). \tag{3}$$

1 Basic Feedback Stabilization

Example 1 : *Consider the system*

$$\dot{x} = ax^2 + u \quad \textit{where `a' is a non-zero constant} \tag{4}$$

we look for a state feedback of the form

$$u = \phi(x)$$

that makes $x = 0$ "asymptotically stable." Consider first setting

$$u = -ax^2 - x \tag{5}$$

Substituting (5) into (4) we obtain

$$\dot{x} = -x$$

which is linear and globally asymptotically stable, as desired. □

Issues: this first solution has 2 problems

(i) It is based on the exact cancelation of the nonlinear term ax^2, thus requiring the exact knowledge of the system parameter(s).

(ii) Canceling "all" nonlinear terms simplifies the analysis but may not be a good idea.

Example 2 : *Consider the system given by*

$$\dot{x} = ax^2 - x^3 + u$$

following the approach in Example 1 we can set

$$u \triangleq u_1 = -ax^2 + x^3 - x$$

which leads to

$$\dot{x} = -x.$$

□

u_1 cancels the terms ax^2 and $-x^3$, which are quite different:

- The term in x^2 is never desirable. It has a destabilizing effect.
- The term in $-x^3$ provides "damping" for x and can be beneficial.
- Cancellation of the term x^3 was achieved by incorporating the term x^3 *in the feedback law.* Leads to very large input values.

Alternate solution: Given the system

$$\dot{x} = f(x, u) \qquad x \in R^n, u \in R, \quad f(0, 0) = 0$$

we proceed to find a feedback law $u = \phi(x)$ such that

$$\dot{x} = f(x, \phi(x)) \tag{6}$$

and $x = 0$ is asymptotically stable. We look for
$V_1 = V_1(x) : D \to R$ satisfying

(i) $V_1(0) = 0$, and $V_1(x)$ is positive definite in $D - \{0\}$.

(ii) There exist $L(x) : D \to R^+$ (positive definite) such that

$$\dot{V}_1(x) = \frac{\partial V_1}{\partial x} f(x, \phi(x)) \ \leq \ -L(x) \quad \forall x \in D.$$

Example 3 : *Consider again the system of example 2.*

$$\dot{x} = ax^2 - x^3 + u$$

defining $V_1(x) = \frac{1}{2}x^2$ *and computing* $\dot{V}_1$*, we obtain*

$$\dot{V}_1 = ax^3 - x^4 + xu.$$

In Example 2 we chose $u = u_1 = -ax^2 + x^3 - x$*. Thus*

$$\dot{V}_1 = ax^3 - x^4 + x(-ax^2 + x^3 - x) = -x^2 \triangleq -L(x).$$

We now modify $L(x)$ as follows:

$$\dot{V}_1 = ax^3 - x^4 + xu \ \leq \ -L(x) \ \triangleq \ -(x^4 + x^2).$$

choose u as,

$$u = -x - ax^2.$$

With this u, we obtain the following feedback system

$$\begin{aligned} \dot{x} &= ax^2 - x^3 + u \\ &= -x - x^3 \end{aligned}$$

which is globally asymptotically stable. □

2 Integrator Backstepping

We consider a system of the form

$$\dot{x} = f(x) + g(x)\xi, \qquad x \in R^n, \xi \in R \tag{7}$$
$$\dot{\xi} = u. \tag{8}$$

We will make the following assumptions (see Figure 5.1(a)):

(i) $f(0) = 0$.

(ii) Viewing the state variable ξ as an independent "input", we assume that there exists a state feedback control law that stabilizes the origin of the subsystem (7)

$$\xi = \phi(x), \qquad \phi(0) = 0$$

and a Lyapunov function $V_1 : D \to R^+$ such that

$$\dot{V}_1(x) = \frac{\partial V_1}{\partial x}[f(x) + g(x) \cdot \phi(x)] \leq -L(x) \leq 0 \quad \forall x \in D$$

where $L(\cdot) : D \to R^+$ is a positive definite function in D.

To stabilize now the system (7)–(8) we proceed as follows:

- Adding and subtracting $g(x)\phi(x)$ to (7) (Figure 1(b)) we obtain the equivalent system

$$\dot{x} = f(x) + g(x)\phi(x) + g(x)[\xi - \phi(x)] \tag{9}$$
$$\dot{\xi} = u. \tag{10}$$

- Define

$$
\begin{aligned}
z &= \xi - \phi(x) && (11)\\
\dot{z} &= \dot{\xi} - \dot{\phi}(x) \;=\; u - \dot{\phi}(x) && (12)
\end{aligned}
$$

where

$$\dot{\phi} = \frac{\partial \phi}{\partial x}\dot{x} = \frac{\partial \phi}{\partial x}[f(x) + g(x)\xi] \tag{13}$$

This change of variables can be seen as "backstepping" $-\phi(x)$ through the integrator (Figure 1(c)). Defining

$$v = \dot{z} \tag{14}$$

the resulting system is (Figure 1(d))

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)\phi(x) + g(x)z && (15)\\
\dot{z} &= v && (16)
\end{aligned}
$$

V is defined as $V = u - \dot{\phi}(x)$ and $\dot{\phi}(x)$ calulated as (13)

Notice that (15)–(16) is equivalent to (7)–(8). To stabilize the system (15)–(16) consider:

$$V = V(x, \xi) = V_1(x) + \frac{1}{2}z^2. \tag{17}$$

$$
\begin{aligned}
\Rightarrow \dot{V} &= \frac{\partial V_1}{\partial x}[f(x) + g(x)\phi(x) + g(x)z] + z\dot{z}\\
&= \frac{\partial V_1}{\partial x}f(x) + \frac{\partial V_1}{\partial x}g(x)\phi(x) + \frac{\partial V_1}{\partial x}g(x)z + zv.
\end{aligned}
$$

We can choose

$$v = -\left(\frac{\partial V_1}{\partial x}g(x) + kz\right), \quad k > 0 \tag{18}$$

Thus

$$
\begin{aligned}
\dot{V} &= \frac{\partial V_1}{\partial x}f(x) + \frac{\partial V_1}{\partial x}g(x)\phi(x) - kz^2\\
&= \frac{\partial V_1}{\partial x}[f(x) + g(x)\phi(x)] - kz^2\\
&\le -L(x) - kz^2. && (19)
\end{aligned}
$$

(19) implies that the origin $x = 0, z = 0$ is asymptotically stable. Since $z = \xi - \phi(x)$ and $\phi(0) = 0$, the origin of the original system $x = 0, \xi = 0$ is also asymptotically stable. The stabilizing state feedback law is given by

$$u = \dot{z} + \dot{\phi} \tag{20}$$

$$u = \frac{\partial \phi}{\partial x}[f(x) + g(x)\xi] - \frac{\partial V_1}{\partial x} g(x) - k[\xi - \phi(x)]. \tag{21}$$

Example 4 : *Consider the following system:*

$$\dot{x}_1 = ax_1^2 - x_1^3 + x_2 \qquad (22)$$
$$\dot{x}_2 = u. \qquad (23)$$

Clearly this system is of the form (15)–(16) with

$$\begin{aligned} x &= x_1 \\ \xi &= x_2 \\ f(x) &= f(x_1) = ax_1^2 - x_1^3 \\ g(x) &= 1 \end{aligned}$$

Step 1: Find $\xi = \phi(x)$ to stabilize the origin $x = 0$. Defining

$$\begin{aligned} V_1(x_1) &= \frac{1}{2}x_1^2 \\ &\Rightarrow \dot{V}_1(x_1) = ax_1^3 - x_1^4 + x_1x_2 \leq -V_a(x_1) \triangleq -(x_1^4 + x_1^2) \end{aligned}$$

choosing

$$x_2 = \phi(x_1) = -x_1 - ax_1^2$$

we obtain

$$\dot{x}_1 = -x_1 - x_1^3.$$

Step 2: To stabilize (22)– (23), we use of the control law (21):

$$\begin{aligned} u &= \frac{\partial \phi}{\partial x}[f(x) + g(x)\xi] - \frac{\partial V_1}{\partial x}g(x) - k[\xi - \phi(x)] \\ &= -(1 + 2ax_1)[ax_1^2 - x_1^3 + x_2] - x_1 - k[x_2 + x_1 + ax_1^2]. \end{aligned}$$

With this control law the origin is globally asymptotically stable. The composite Lyapunov function is

$$\begin{aligned} V &= V_1 + \frac{1}{2}z^2 = \frac{1}{2}x_1^2 + \frac{1}{2}[x_2 - \phi(x_1)]^2 \\ &= \frac{1}{2}x_1^2 + \frac{1}{2}[x_2 - x_1 + ax_1^2]^2. \end{aligned}$$

3 Chain of Integrators

Consider now a system of the form

$$\begin{aligned}
\dot{x} &= f(x) + g(x)\xi_1 \\
\dot{\xi}_1 &= \xi_2 \\
&\vdots \\
\dot{\xi}_{k-1} &= \xi_k \\
\dot{\xi}_k &= u
\end{aligned}$$

For simplicity we focus on the third order system

$$\dot{x} = f(x) + g(x)\xi_1 \tag{24}$$

$$\dot{\xi}_1 = \xi_2 \tag{25}$$

$$\dot{\xi}_2 = u \tag{26}$$

To stabilize the origin we proceed as follows: we consider the first "subsystem" (24). Assume that $\xi_1 = \phi(x_1)$ is a stabilizing control law for the system with Lyapunov function V_1. Consider now the first 2 subsystems:

$$\dot{x} = f(x) + g(x)\xi_1 \tag{27}$$

$$\dot{\xi}_1 = \xi_2 \tag{28}$$

We can stabilize this second order system using backstepping. Using the control law (21) and associated Lyapunov function V_2:

$$\begin{aligned}
\xi_2 \;=\; \phi(x,\xi_1) &= \tfrac{\partial\phi(x)}{\partial x}[f(x)+g(x)\xi_1] \\
&\quad -\tfrac{\partial V_1}{\partial x}g(x) - k[\xi_1 - \phi(x)] \quad , \qquad k>0 \\
V_2 &= V_1 + \tfrac{1}{2}[\xi_1 - \phi(x)]^2
\end{aligned}$$

We now iterate view the third-order system as a more general version of (7)–(8) with

$$x = \begin{bmatrix} x \\ \xi_1 \end{bmatrix}, \quad \xi = \xi_2, \quad f = \begin{bmatrix} f(x)+g(x)\xi_1 \\ 0 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Applying the backstepping algorithm once again, we obtain:

$$\begin{aligned}
u &= \frac{\partial\phi(x)}{\partial x}\dot{x} - \frac{\partial V_2}{\partial x}g(x) - \; k[\xi_2 - \phi(x)], \quad k>0 \\
&= \left[\frac{\partial\phi(x,\xi_1)}{\partial x}, \frac{\partial\phi(x,\xi_1)}{\partial \xi_1}\right] [\dot{x}, \dot{\xi}_1]^T - \left[\frac{\partial V_2}{\partial x}, \frac{\partial V_2}{\partial \xi_1}\right] [0 \;,\; 1]^T + \\
&\quad -k[\xi_2 - \phi(x,\xi_1)], \quad k>0
\end{aligned}$$

or

$$\begin{aligned} u &= \frac{\partial\phi(x,\xi_1)}{\partial x}[f(x)+g(x)\xi_1] + \frac{\partial\phi(x,\xi_1)}{\partial\xi_1}\xi_2 - \frac{\partial V_2}{\partial\xi_1} + \\ &\quad -k[\xi_2-\phi(x,\xi_1)], \quad k>0. \end{aligned}$$

The composite Lyapunov function is

$$\begin{aligned} V &= V_2 + \frac{1}{2}[\xi_2-\phi(x,\xi_1)]^2 \\ &= V_1 + \frac{1}{2}[\xi_1-\phi(x)]^2 + \frac{1}{2}[\xi_2-\phi(x,\xi_1)]^2. \end{aligned}$$

Example 5 :

$$\begin{aligned}\dot{x}_1 &= ax_1^2 - x_1^3 + x_2 \\ \dot{x}_2 &= u\end{aligned}$$

Substitute $x_2 = x_{2d} + \eta_2$ *to the system equation, get*

$$\begin{aligned}\dot{x}_1 &= ax_1^2 - x_1^3 + x_{2d} + \eta_2 \\ \dot{\eta}_2 &= -\dot{x}_{2d} + u\end{aligned}$$

Design x_{2d} *as* $x_{2d} = ax_1^2 - x_1$, *thus the first system equation can be written as:*

$$\dot{x}_1 = -x_1 - x_1^3 + \eta_2$$

Since x_{2d} *is designed as* $x_{2d} = ax_1^2 - x_1$, *then*

$$\begin{aligned}\dot{x}_{2d} &= \frac{d}{dt}(ax_1^2 - x_1) \\ &= (2x_1a + x_1)\dot{x}_1 \\ &= (2x_1a + x_1)(ax_1^2 - x_1^3 + x_2)\end{aligned}$$

use $\kappa(x_1, x_2)$ *to represent* $\dot{x}_{2d}$, *then the second system equation can be written as:*

$$\dot{\eta}_2 = -\kappa(x_1, x_2) + u$$

Design u *as*

$$u = -\kappa(x_1, x_2) - \eta_2 + u_{aux}$$

The second equation of the system becomes:

$$\dot{\eta}_2 = -\eta_2 + u_{aux}$$

Now for the following system:

$$\begin{aligned}\dot{x}_1 &= -x_1 - x_1^3 \\ \dot{\eta}_2 &= -\eta_2 + u_{aux}\end{aligned}$$

Using the following Lyapunov function candidate: $V = \frac{1}{2}x_1^2 + \frac{1}{2}\eta_2^2$*, we have:*

$$\begin{aligned}
\dot{V} &= x_1(-x_1 - x_1^3 + \eta_2) + \eta_2(-\eta_2 + u_{aux}) \\
\dot{V} &\leq -x_1^2 + x_1\eta_2 - \eta_2^2 + \eta_2 u_{aux}
\end{aligned}$$

if we let $u_{aux} = -x_1$*, we can have*

$$\dot{V} \leq -x_1^2 - \eta_2^2$$

Therefore the stabalizing control law can be:

$$\begin{aligned}
u &= -\kappa(x_1, x_2) - \eta_2 + u_{aux} \\
&= -(2x_1 a + x_1)(ax_1^2 - x_1^3 + x_2) - (\dot{x}_1 + x_1 + x_1^3) - x_1 \\
&= (2a^2 + a)x_1^3 - (2a + 1)x_1^4 + (2a + 1)x_1 x_2
\end{aligned}$$

Example 6 *: Consider the following system, :*

$$\begin{aligned}
\dot{x}_1 &= ax_1^2 + x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= u.
\end{aligned}$$

Step 1: Consider the first equation $\dot{x}_1 = ax_1^2 + \phi(x_1)$*. Using* $V_1 = \frac{1}{2}x_1^2$*, it is immediate that* $\phi(x_1) = -x_1 - ax_1^2$ *stabilizes the origin.*
Step 2: Consider the first two subsystems. We propose the stabilizing law (with $k > 0$*) and associated Lyapunov function:*

$$\phi(x_1, x_2)(= x_3) = \frac{\partial \phi(x_1)}{\partial x_1}[f(x_1) + g(x_1)x_2] - \frac{\partial V_1}{\partial x_1}g(x_1) - k[x_2 - \phi(x_1)]$$

$$\begin{aligned}
V_2 &= V_1 + \frac{1}{2}z^2 = V_1 + \frac{1}{2}[x_2 - \phi(x_1)] \\
&= V_1 + \frac{1}{2}[x_2 + x_1 + ax_1^2]^2.
\end{aligned}$$

In our case, setting $k = 1$*,*

$$\begin{aligned}
\frac{\partial \phi(x_1)}{\partial x_1} &= -(1 + 2ax_1) \\
\frac{\partial V_1}{\partial x_1} &= x_1
\end{aligned}$$

$$\Rightarrow \quad \phi(x_1, x_2) = -(1 + 2ax_1)[ax_1^2 + x_2] - x_1 - [x_2 + x_1 + ax_1^2]$$

Step 3: Consider the third order system with

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \xi = x_3, \quad f = \begin{bmatrix} f(x_1) + g(x_1)x_2 \\ 0 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

From the results in the previous section we have that

$$\begin{array}{rcl} u & = & \frac{\partial \phi(x_1,x_2)}{\partial x_1}[f(x_1) + g(x_1)x_2] + \quad \frac{\partial \phi(x_1,x_2)}{\partial x_2} x_3 - \frac{\partial V_2}{\partial x_2} + \\ & & -k[x_3 - \phi(x_1, x_2)] \qquad , \quad k > 0 \end{array}$$

is a stabilizing control law with associated Lyapunov function

$$V = V_2 + \frac{1}{2}[x_3 - \phi(x_1, x_2]^2$$

□

4 Strict Feedback Systems

Consider now "strict feedback systems" of the form

$$\begin{aligned}
\dot{x} &= f(x) + g(x)\xi_1 \\
\dot{\xi}_1 &= f_1(x,\xi_1) + g_1(x,\xi_1)\xi_2 \\
\dot{\xi}_2 &= f_2(x,\xi_1,\xi_2) + g_2(x,\xi_1,\xi_2)\xi_3 \\
&\vdots \\
\dot{\xi}_{k-1} &= f_{k-1}(x,\xi_1,\xi_2,\cdots,\xi_{k-1}) + g_{k-1}(x,\xi_1,\xi_2,\cdots,\xi_{k-1})\xi_k \\
\dot{\xi}_k &= f_k(x,\xi_1,\xi_2,\cdots,\xi_k) + g_k(x,\xi_1,\xi_2,\cdots,\xi_k)u
\end{aligned}$$

also called *triangular systems.* Considering first the special case:

$$\dot{x} = f(x) + g(x)\xi \tag{29}$$

$$\dot{\xi} = f_a(x,\xi) + g_a(x,\xi)u. \tag{30}$$

If $g_a(x,\xi) \neq 0$ over the domain of interest, then we can define

$$u = \phi(x,\xi) \triangleq \frac{1}{g_a(x,\xi)}[u_1 - f_a(x,\xi)]. \tag{31}$$

Substituting (31) into (30) we obtain the modified system

$$\dot{x} = f(x) + g(x)\xi \tag{32}$$

$$\dot{\xi} = u_1 \tag{33}$$

which is of the form (7)–(8). The stabilizing control law and associated Lyapunov function are thus:

$$\begin{aligned}
u &= \phi_1(x,\xi) = \frac{1}{g_a(x,\xi)}\left\{\frac{\partial\phi}{\partial x}[f(x)+g(x)\xi]\right. \qquad (34)\\
&\qquad \left. -\frac{\partial V_1}{\partial x}g(x) - k_1[\xi-\phi(x)] - f_a(x,\xi)\right\}, \quad k_1 > 0
\end{aligned}$$

$$V_2 = V_2(x,\xi) = V_1(x) + \frac{1}{2}[\xi - \phi(x)]^2. \tag{35}$$

Considering now the system

$$\begin{aligned}
\dot{x} &= f(x) + g(x)\xi_1 \\
\dot{\xi}_1 &= f_1(x,\xi_1) + g_1(x,\xi_1)\xi_2 \\
\dot{\xi}_2 &= f_2(x,\xi_1,\xi_2) + g_2(x,\xi_1,\xi_2)\xi_3
\end{aligned}$$

which can be seen as a special case of (29)–(30) with

$$\begin{aligned} x &= \begin{bmatrix} x \\ \xi_1 \end{bmatrix}, \quad \xi = \xi_2, \quad u = \xi_3, f = \begin{bmatrix} f + g\ \xi_1 \\ f_1 \end{bmatrix}, g = \begin{bmatrix} 0 \\ g_1 \end{bmatrix}, \\ f_a &= f_2,\ g_a = g_2. \end{aligned}$$

The stabilizing control law and associated Lyapunov function for this systems are as follows:

$$\begin{aligned} \phi_2(x, \xi_1, \xi_2) &= \frac{1}{g_2}\left\{ \frac{\partial \phi_1}{\partial x}(f + g\xi_1) + \frac{\partial \phi_1}{\partial \xi_1}(f_1(x) + g_1(x)\xi_2) + \right. \\ &\qquad \left. - \frac{\partial V_2}{\partial \xi_1} g_1 - k_2[\xi_2 - \phi_1] - f_2 \right\}, \quad k_2 > 0 \end{aligned} \tag{36}$$

$$V_3(x, \xi_1, \xi_2) = V_2(x) + \frac{1}{2}[\xi_2 - \phi_1(x, \xi_1)]^2. \tag{37}$$

Example 7 : *Consider the following systems:*

$$\begin{aligned}\dot{x}_1 &= ax_1^2 - x_1 + x_1^2 x_2 \\ \dot{x}_2 &= x_1 + x_2 + (1 + x_2^2)u.\end{aligned}$$

We begin by stabilizing the x subsystem. Using $V_1 = 1/2x_1^2$ we have that

$$\begin{aligned}\dot{V}_1 &= x_1[ax_1^2 - x_1 + x_1^2 x_2] \\ &= ax_1^3 - x_1^2 + x_1^3 x_2.\end{aligned}$$

Thus, $x_2 = \phi(x_1) = -(x_1 + a)$ results in

$$\dot{V}_1 = -(x_1^2 + x_1^4)$$

which shows that the x_1 system is asymptotically stable. It then follows by (34)–(35) that a stabilizing control law for the second-order system and the corresponding Lyapunov function are given by

$$\begin{aligned}u &= \phi_1(x_1, x_2) = \frac{1}{(1 + x_2^2)}\{-(1 + a)[ax_1^2 - x_1^3 + x_1^2 x_2] - x_1^3 + \\ &\quad -k_1[x_2 + x_1 + a] - (x_1 + x_2)\} \ , \quad k_1 > 0 \\ V_2 &= \frac{1}{2}x_1^2 + \frac{1}{2}[x_1 + x_2 + a]^2.\end{aligned}$$

□