Chapter 6: Input–Output Stability

1 Function Spaces

In the following definition, we consider a function $u : R^+ \to R^q$, i.e., u is of the form:

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdots \\ u_q(t) \end{bmatrix}$$

Definition 1 *: (The Space $\mathcal{L}_2$) The space $\mathcal{L}_2$ consists of all piecewise continuous functions $u : R^+ \to R^q$ satisfying*

$$\|u\|_{\mathcal{L}_2} \triangleq \sqrt{\int_0^\infty [|u_1|^2 + |u_2|^2 + \cdots + |u_q|^2] \, \mathrm{d}t} \;\; < \;\; \infty. \tag{1}$$

The norm $\|u\|_{\mathcal{L}_2}$ is the so-called $\mathcal{L}_2$ norm of the function u.

Definition 2 *: (The Space $\mathcal{L}_\infty$) The space $\mathcal{L}_\infty$ consists of all piecewise continuous functions $u : R^+ \to R^q$ satisfying*

$$\|u\|_{\mathcal{L}_\infty} \triangleq \sup_{t \in R^+} \|u(t)\|_\infty < \infty. \tag{2}$$

$\mathcal{L}_2$ and $\mathcal{L}_\infty$ are examples of the so-called $\mathcal{L}_p$ spaces: given $p : 1 \leq p < \infty$ the space $\mathcal{L}_p$ consists of all piecewise continuous functions satisfying

$$\|u\|_{\mathcal{L}_p} \triangleq \left(\int_0^\infty [|u_1|^p + |u_2|^p + \cdots + |u_q|^p] \, dt \right)^{1/p} \;\; < \;\; \infty. \tag{3}$$

1.1 Extended Spaces

Definition 3 *: Let $u \in \mathcal{X}$. We define the truncation operator $P_T : \mathcal{X} \to \mathcal{X}$ by*

$$(P_T u)(t) \equiv u_T(t) \triangleq \begin{cases} u(t), & t \leq T \\ \\ 0, & t > T \end{cases} \qquad t, T \in R^+ \tag{4}$$

Example 1 *: Consider the function $u : R^+ \to R^+$ defined by $u(t) = t^2$. The truncation of $u(t)$ is the following function:*

$$u_T(t) = \begin{cases} t^2, & 0 \leq t \leq T \\ 0, & t > T \end{cases}$$

□

Definition 4 *: The extension of the space $\mathcal{X}$, denoted $\mathcal{X}e$ is defined as the space consisting of all functions whose truncation belongs to $\mathcal{X}$*

Example 2 *: Let the space of functions $\mathcal{X} = \mathcal{L}_\infty$ and Consider the function $x(t) = t$. Thus $x_T \in \mathcal{X}e \forall T \in R^+$. However $x \not\in \mathcal{X}$ since $\lim_{T \to \infty} |x_T| = \infty$.* □

2 Input–Output Stability

We start with a precise definition of the notion of system.

Definition 5 *: A system, in input-output sense, is a mapping $H : \mathcal{X}e \to \mathcal{X}e$ that satisfies the so-called causality condition:*

$$[Hu(\cdot)]_T = [Hu_T(\cdot)]_T \qquad \forall u \in \mathcal{X}e \text{ and } \forall T \in R. \tag{5}$$

Condition (5) states that the past and present outputs do not depend on future inputs. To visualize, imagine the following experiments (Figures 1 and 2):

(1) We apply $u(t)$, we find $y(t) = Hu(t)$, and from here $y_T(t) = [Hu(t)]_T$. Clearly $y_T = [Hu(t)]_T(t)$is the left-hand side of (5). See Figure 2 (a)-(c).

(2) We compute $\bar{u} = u_T(t)$ from the $u(t)$ used above, and repeat the procedure used in the first experiment. Namely, we compute the output $\bar{y}(t) = H\bar{u}(t) = Hu_T(t)$ to the input $\bar{u}(t) = u_T(t)$. Finally compute $\bar{y}_T = [Hu_T(t)]_T$ from $\bar{y}$. This is the the right-hand side of equation (5). See Figure 2 (d)-(f).

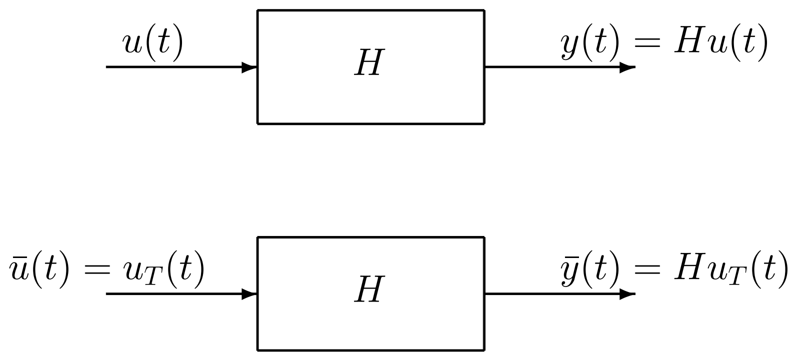

Figure 1: *Experiment 1*: input $u(t)$ applied to system H. *Experiment 2*: input $\bar{u}(t) = u_T(t)$ applied to system H.

We may now state the definition of input–output stability.

Definition 6 *: A system $H : \mathcal{X}e \to \mathcal{X}e$ is input-output $\mathcal{X}$-stable if whenever the input belongs to the space $\mathcal{X}$, the output is once again in $\mathcal{X}$. In other words, H is $\mathcal{X}$-stable if Hx is in $\mathcal{X}$ whenever u in $\mathcal{X}$.*

Definition 7 *: A system $H : \mathcal{X}e \to \mathcal{X}e$ is said to have a finite gain if there exists a constant $\gamma(H) < \infty$ called the gain of H, and a constant $\beta \in R^+$ such that*

$$\|(Hu)_T\|_{\mathcal{X}} \leq \gamma(H)\ \|u_T\|_{\mathcal{X}} + \beta. \tag{6}$$

If the system H satisfies the condition

$$Hu = 0 \quad whenever \;\; u = 0$$

then the gain $\gamma(H)$ can be calculated as follows

$$\gamma(H) = \sup \frac{\|(Hu)_T\|_{\mathcal{X}}}{\|u_T\|_{\mathcal{X}}} \tag{7}$$

where the supremum is taken over all $u \in \mathcal{X}e$ and all T in R^+ for which $u_T \neq 0$.

Example 3 *: Let $\mathcal{X} = \mathcal{L}_\infty$, and consider the nonlinear operator $N(\cdot)$ defined by the graph in the plane shown in Figure (3), and notice that $N(0) = 0$. The gain $\gamma(H)$ is easily determined from the slope of the graph of N.*

$$\gamma(H) = \sup \frac{\|(Hu)_T\|_{\mathcal{L}_\infty}}{\|u_T\|_{\mathcal{L}_\infty}} \ = \ 1.$$

□

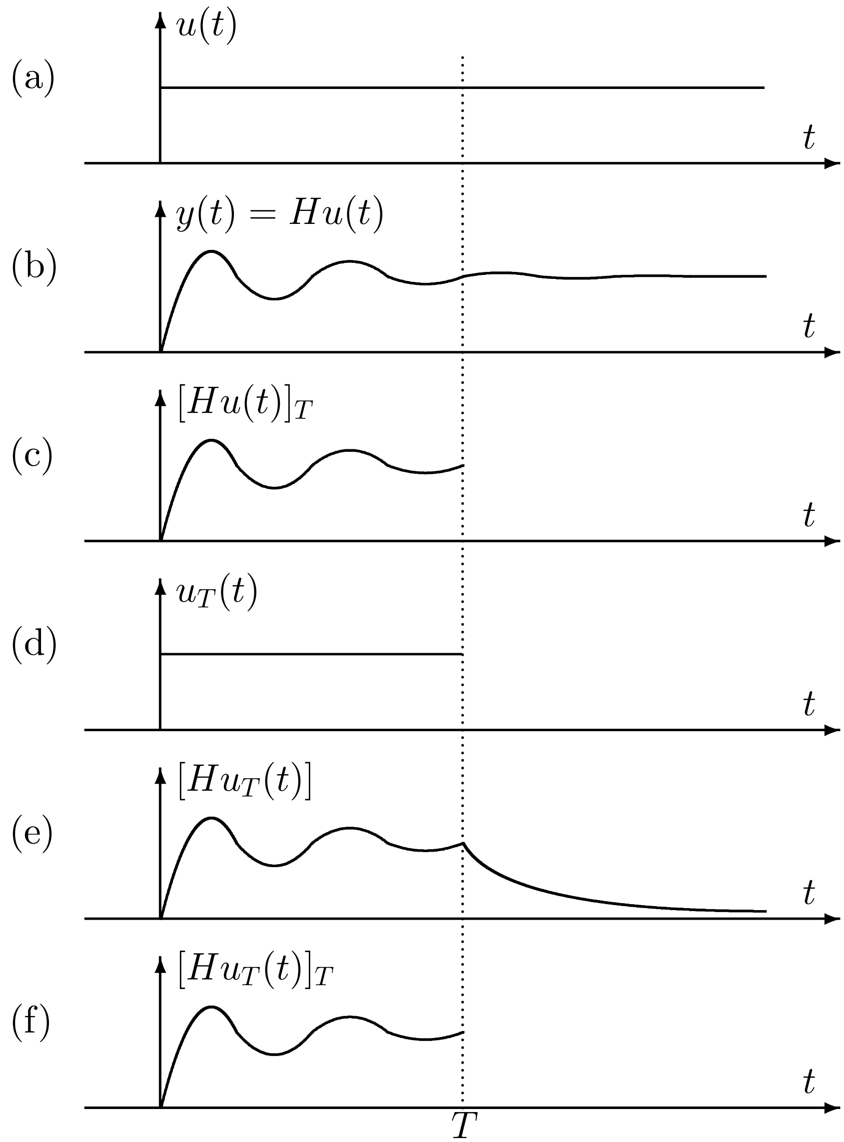

Figure 2: Causal systems: (a) input $u(t)$; (b) the response $y(t) = Hu(t)$; (c) truncation of the response $y(t)$. Notice that this figure corresponds to the left-hand side of equation (5); (d) truncation of the function $u(t)$; (e) response of the system when the input is the truncated input $u_T(t)$; (f) truncation of the system response in part (e). Notice that this figure corresponds to the right-hand side of equation (5).

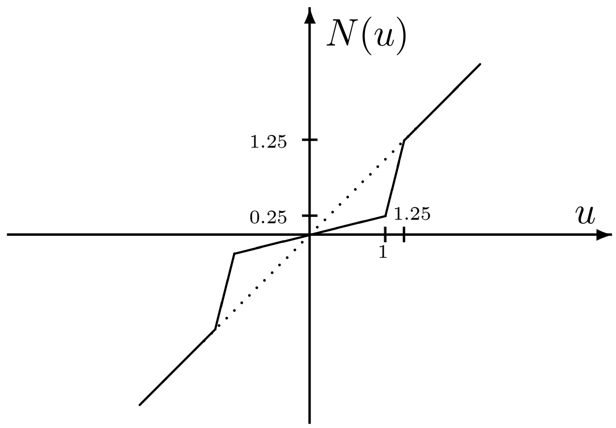

Figure 3: Static nonlinearity $N(\cdot)$.

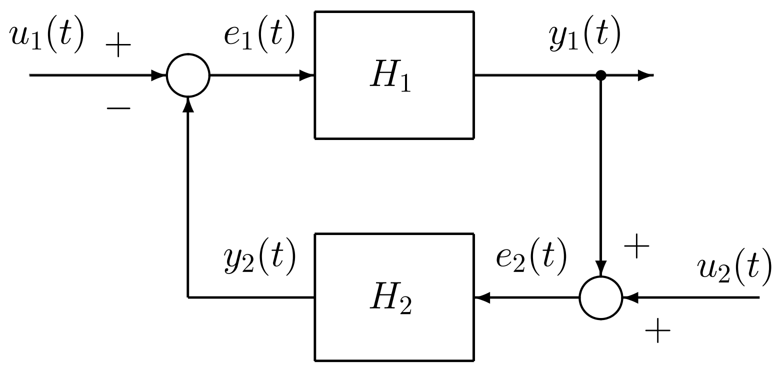

Figure 4: The Feedback System S.

3 The Small Gain Theorem

Definition 8 *: We will denote by feedback system to the interconnection of the subsystems H_1 and $H_2 : \mathcal{X}_e \to \mathcal{X}_e$ that satisfies the following assumptions:*

(i) $e_1,\ e_2,\ y_1,$ *and* $y_2 \in \mathcal{X}_e$ *for all pairs of inputs* $u_1, u_2 \in \mathcal{X}_e$*.*

(ii) The following equations are satisfied for all $u_1,\ u_2 \in \mathcal{X}_e$*:*

$$e_1 = u_1 - H_2 e_2 \tag{8}$$

$$e_2 = u_2 + H_1 e_1. \tag{9}$$

It is immediate that equations (8) and (9) can be represented graphically as shown in Figure (4).

Theorem 1 *: Consider the feedback interconnection of the systems H_1 and $H_2 : \mathcal{X}e \rightarrow \mathcal{X}e$. Then, if $\gamma(H_1)\gamma(H_2) < 1$, the feedback system is input–output–stable.*

Proof: For simplicity we assume that $\beta = 0$. We must show that $u_1, u_2 \in \mathcal{X}$ imply that e_1, e_2, y_1 and y_2 are also in $\mathcal{X}$. Truncating (8) and (9), we have

$$\begin{aligned} e_{1T} &= u_{1T} - (H_2 e_2)_T & (10) \\ e_{2T} &= u_{2T} + (H_1 e_1)_T. & (11) \end{aligned}$$

Thus,

$$\begin{aligned} \|e_{1T}\| &\leq \|u_{1T}\| + \|(H_2 e_2)_T\| \leq \|u_{1T}\| + \gamma(H_2)\|e_{2T}\| & (12) \\ \|e_{2T}\| &\leq \|u_{2T}\| + \|(H_1 e_1)_T\| \leq \|u_{2T}\| + \gamma(H_1)\|e_{1T}\|. & (13) \end{aligned}$$

Substituting (13) in (12) we obtain

$$\begin{aligned} \|e_{1T}\| &\leq \|u_{1T}\| + \gamma(H_2)\{\|u_{2T}\| + \gamma(H_1)\|e_{1T}\|\} \\ &\leq \|u_{1T}\| + \gamma(H_2)\|u_{2T}\| + \gamma(H_1)\gamma(H_2)\|e_{1T}\| \end{aligned}$$

$$\Rightarrow \quad [1 - \gamma(H_1)\gamma(H_2)]\|e_{1T}\| \leq \|u_{1T}\| + \gamma(H_2)\|u_{2T}\| \tag{14}$$

and since, by assumption, $\gamma(H_1)\gamma(H_2) < 1$,

$$\|e_{1T}\| \leq [1 - \gamma(H_1)\gamma(H_2)]^{-1}\{\|u_{1T}\| + \gamma(H_2)\|u_{2T}\|\}. \tag{15}$$

Similarly

$$\|e_{2T}\| \leq [1 - \gamma(H_1)\gamma(H_2)]^{-1}\{\|u_{2T}\| + \gamma(H_1)\|u_{1T}\|\}. \tag{16}$$

If, in addition, u_1 and u_2 are in $\mathcal{X}$ we can take limits as $T \rightarrow \infty$:

$$\begin{aligned} \|e_1\| &\leq [1 - \gamma(H_1)\gamma(H_2)]^{-1}\{\|u_1\| + \gamma(H_2)\|u_2\|\} & (17) \\ \|e_2\| &\leq [1 - \gamma(H_1)\gamma(H_2)]^{-1}\{\|u_2\| + \gamma(H_1)\|u_1\|\}. & (18) \end{aligned}$$

It follows that e_1 and e_2 are also in $\mathcal{X}$. Finally,

$$\|(H_i e_i)_T\| \leq \gamma(H_i)\|e_{iT}\|, \quad i = 1, 2 \tag{19}$$

$\Rightarrow y_i \in \mathcal{X}$. $\square$