Chapter 7: Input-to-State Stability

1 Motivation

Consider again the system

$$\dot{x} = f(x, u) \tag{1}$$

Assuming that $\dot{x} = f(x, 0)$ has a uniformly asymptotically stable equilibrium point at the origin, we study what happens when $u \neq 0$.

Example 1 : *Consider the following first-order nonlinear system:*

$$\dot{x} = -x + (x + x^3)u.$$

If $u = 0$, we obtain $\dot{x} = -x$, (asymptotically stable equilibrium point). When $u(t) = 1$, however, we obtain $\dot{x} = x^3$, which results in an unbounded trajectory for any initial condition. □

⇒ Asymptotic stability of $x = 0$ does not say much about the forced system.

2 Definitions

The notion of input-to-state stability (ISS) attepts to capture the notion of "bounded input–bounded state".

Definition 1 : *The system (1) is said to be locally input-to-state-stable (ISS) if there exist a $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ and constants $k_1, k_2 \in R^+$ such that*

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\|u_T(\cdot)\|_{\mathcal{L}_\infty}), \qquad \forall t \geq 0, \quad 0 \leq T \leq t \tag{2}$$

for all $x_0 \in D$ and $u \in D_u$ satisfying : $\|x_0\| < k_1$ and $\sup_{t>0} \|u_T(t)\| = \|u_T\|_{\mathcal{L}_\infty} < k_2, \quad 0 \leq T \leq t$. It is said to be input-to-state stable, or globally ISS if $D = R^n$, $D_u = R^m$ and (2) is satisfied for any initial state and any bounded input u.

Implications (Assume that $\dot{x} = f(x, u)$ is ISS)

- *Unforced systems*: consider $\dot{x} = f(x, 0)$. The response of (1) with initial state x_0 satisfies

$$\|x(t)\| \leq \beta(\|x_0\|, t) \qquad \forall t \geq 0, \|x_0\| < k_1,$$

  $\Rightarrow$ $x = 0$ is uniformly asymptotically stable.

- Interpretation: if $\|u\|_\infty < \delta$, trajectories remain bounded by the ball of radius $\beta(\|x_0\|, t) + \gamma(\delta)$, i.e.,

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\delta).$$

  As t increases, $\beta(\|x_0\|, t) \to 0$ and trajectories approach the ball of radius $\gamma(\delta)$, i.e.,

$$\begin{aligned} \lim_{t \to \infty} \|x(t)\| &= L \\ L &\leq \gamma(\delta) \end{aligned}$$

  $\gamma(\cdot)$ is called the ultimate bound of the system (1).

- *Alternative Definition*: A variation of Definition 1 is to replace equation (2) with the following equation:

$$\|x(t)\| \leq \max\{\beta(\|x_0\|, t), \gamma(\|u_T(\cdot)\|_{\mathcal{L}_\infty})\}, \quad \forall t \geq 0, \ \ 0 \leq T \leq t. \qquad (3)$$

Definition 2 : *A continuously differentiable function $V : D \to R$ is said to be an ISS Local Lyapunov function on D for the system (1) if there exist class $\mathcal{K}$ functions α_1, α_2, α_3, and $\mathcal{X}$ such that:*

$$\alpha_1(\|x\|) \ \leq \ V(x(t)) \ \leq \ \alpha_2(\|x\|) \qquad \forall x \in D, \ t > 0 \qquad (4)$$

$$\frac{\partial V(x)}{\partial x} f(x, u) \ \leq \ -\alpha_3(\|x\|) \quad u \in D_u : \|x\| \geq \mathcal{X}(\|u\|). \qquad (5)$$

V is said to be an ISS Global Lyapunov function if $D = R^n$, $D_u = R^m$, and $\alpha_1, \ \alpha_2, \ \alpha_3 \in \mathcal{K}_\infty$.

Remarks: this means that V is an ISS Lyapunov function if

(a) It is positive definite in D.

(b) It is negative definite along the trajectories of (1) *whenever the trajectories are outside of the ball defined by $\|x^*\| = \mathcal{X}(\|u\|)$.*

3 Input-to-State Stability (ISS) Theorems

Theorem 1 *: (Local ISS Theorem) Consider the system (1) and let $V : D \to R$ be an ISS Lyapunov function for this system. Then (1) is input-to-state-stable according to Definition 1 with*

$$\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \mathcal{X} \tag{6}$$

$$k_1 = \alpha_2^{-1}(\alpha_1(r)) \tag{7}$$

$$k_2 = \mathcal{X}^{-1}(\min\{k_1, \mathcal{X}(r_u)\}). \tag{8}$$

$$i.e\ \|x\| < r, \quad \|u\| < r_u \tag{9}$$

Theorem 2 *: (Global ISS Theorem) If the preceeding conditions are satisfied with $D = R^n$ and $D_u = R^m$, and if $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, then the system (1) is globally input-to-state stable.*

Example 2 *: Consider the following system:*

$$\dot{x} = -ax^3 + u \qquad a > 0.$$

We propose the ISS Lyapunov function candidate $V(x) = \frac{1}{2}x^2$*. This* V *is positive definite and satisfies (4) with*

$$\alpha_1(\|x\|) = \alpha_2(\|x\|) = \frac{1}{2}x^2.$$

We have

$$\dot{V} = -x(ax^3 - u). \tag{10}$$

We need to find $\alpha_3(\cdot)$ *and* $\mathcal{X}(\cdot) \in \mathcal{K}$ *such that* $\dot{V}(x) \leq -\alpha_3(\|x\|)$*, whenever* $\|x\| \geq \mathcal{X}(\|u\|)$*. Let* $\theta : 0 < \theta < 1$

$$\begin{aligned} \dot{V} &= -ax^4 + a\theta x^4 - a\theta x^4 + xu \\ &= -a(1-\theta)x^4 - x(a\theta x^3 - u) \\ &\leq -a(1-\theta)x^4 = -\alpha_3(\|x\|) \end{aligned}$$

provided that

$$x(a\theta x^3 - u) > 0.$$

This will be the case, provided that

$$a\theta|x|^3 > |u|$$

or, equivalently

$$\begin{aligned} |x| &> \left(\frac{|u|}{a\theta}\right)^{1/3} \\ \chi(\|u\|) &= \left(\frac{|u|}{a\theta}\right)^{1/3} \end{aligned}$$

It follows that the system is globally input-to-state-stable with $\gamma(u) = \left(\frac{|u|}{a\theta}\right)^{1/3}$.

$\square$

Example 3 *: Now consider the following system, which is a slightly modified version of the one in Example 2:*

$$\dot{x} = -ax^3 + x^2 u \qquad a > 0$$

Using the same ISS Lyapunov function candidate used in Example 2, we have that

$$\begin{aligned}
\dot{V} &= -ax^4 + x^3 u \\
&= -ax^4 + a\theta x^4 - a\theta x^4 + x^3 u \quad 0 < \theta < 1 \\
&= -a(1-\theta)x^4 - x^3(a\theta x - u) \\
&\leq -a(1-\theta)x^4, \qquad \textit{provided}
\end{aligned}$$

$$\begin{aligned}
x^3(a\theta x - u) &> 0 \quad \textit{or}, \\
|x| &> \frac{|u|}{a\theta}.
\end{aligned}$$

Thus, the system is globally input-to-state stable with $\gamma(u) = \frac{|u|}{a\theta}$.

□

4 Input-to-State Stability Revisited

Theorem 3 *: A continuous function $V : D \to R$ is an ISS Lyapunov function on D for the system (1) if and only if there exist class $\mathcal{K}$ functions α_1, α_2, α_3, and σ such that the following two conditions are satisfied:*

$$\alpha_1(\|x\|) \ \leq \ V(x(t)) \ \leq \ \alpha_2(\|x\|) \qquad \forall x \in D, \ t > 0 \tag{11}$$

$$\frac{\partial V(x)}{\partial x} f(x,u) \ \leq \ -\alpha_3(\|x\|) + \sigma(\|u\|) \quad \forall x \in D, u \in D_u \tag{12}$$

V is an ISS Global Lyapunov function if $D = R^n$, $D_u = R^m$, and $\alpha_1, \alpha_2, \alpha_3$, and $\sigma \in \mathcal{K}_\infty$.

Remarks: Notice that, given $r_u > 0$, there exist points $x \in R^n$ such that

$$\alpha_3(\|x\|) = \sigma(r_u).$$

This implies that $\exists d \in R^+$ such that

$$\alpha_3(d) = \sigma(r_u), \quad \textit{or}$$

$$d = \alpha^{-1}(\sigma(r_u)).$$

Denoting $B_d = \{x \in R^n : \|x\| \leq d\}$, we have that for any $\|x\| > d$ and any $u : \|u\|_{\mathcal{L}_\infty} < r_u$:

$$\frac{\partial V}{\partial x} f(x,u) \leq -\alpha(\|x\|) + \sigma(\|u\|) \leq -\alpha(\|d\|) + \sigma(\|u\|_{\mathcal{L}_\infty}).$$

Thus, the trajectory $x(t)$ resulting from an input $u(t) : \|u\|_{\mathcal{L}_\infty} < r_u$ will eventually enter the region

$$\Omega_d = \max_{\|x\| \leq d} V(x).$$

Once inside this region, it is trapped inside Ω_d, because of the condition on $\dot{V}$.

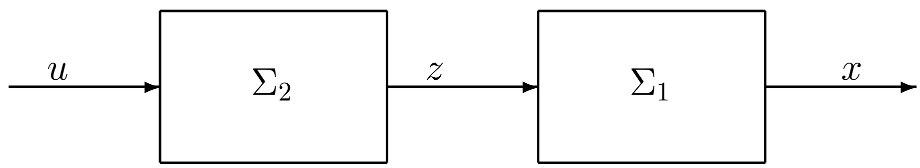

Figure 1: Cascade connection of ISS systems.

5 Cascade-Connected Systems

Throughout this section we consider the composite system shown in Figure 1, where Σ_1 and Σ_2 are given by

$$\begin{aligned} \Sigma_1 : \qquad \dot{x} &= f(x,z) && (13) \\ \Sigma_2 : \qquad \dot{z} &= g(z,u) && (14) \end{aligned}$$

where Σ_2 is the system with input u and state z. The state of Σ_2 serves as input to the system Σ_1.

Theorem 4 *: Consider the cascade interconnection of the systems Σ_1 and Σ_2. If both systems are input-to-state-stable, then the composite system Σ*

$$\Sigma : u \to \begin{bmatrix} x \\ z \end{bmatrix}$$

is input-to-state-stable.