Chapter 8: Passivity

1 Power and Energy: Passive Systems

<u>Notation</u>:

$$\begin{aligned} p(\cdot) &: \text{power} \\ w(\cdot) &: \text{energy} \\ w(t) &= \int_{t_0}^{t} p(t)\, dt. \end{aligned}$$

In particular, in electric circuits, from where we will draw our inspiration, we have:

$$p(t) = v(t)i(t)$$

with the following convention:

(i) If $w(t) > 0$, the box absorbs energy (example, resistence).

(ii) If $w(t) < 0$, the box delivers energy (example, a battery, with negative voltage with respect to the polarity indicated in Figure 8.1).

In circuit theory, elements that do not generate their own energy are called *passive*, i.e., a circuit element is passive if

$$\int_{-\infty}^{t} v(t)i(t)\,dt \geq 0. \qquad (1)$$

Example 1 *: (Resistance)*

$$\begin{aligned} v(t) &= V, \\ i(t) &= \frac{V}{R} \\ p(t) &= \frac{V^2}{R} > 0 \end{aligned}$$

Thus a resistance "absorbs" energy at a rate V^2/R and is a passive element.

Passive elements have some important properties.

- *passivity* has important implication on *stability*, as we will see.
- passivity is a *generic property* of a class of systems, and does not depend on the particular value of the system elements (this is useful).

Problem: we need to generalize the passivity concepts to systems other than electric circuits.

2 Definitions

Definition 1 *: A real inner product space $\mathcal{X}$, is a real linear space with a function $\langle x, y \rangle$ on $\mathcal{X} \times \mathcal{X} \to R$ that satisfies:*

(i) $\langle x, y \rangle = \langle y, x \rangle$.

(ii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y + z \rangle \quad \forall x, y, z \in \mathcal{X}$.

(iii) $\langle \alpha x, y \rangle = \alpha \langle y, x \rangle \quad \forall x, y \in \mathcal{X}, \forall \alpha \in R$.

(iv) $\langle x, x \rangle \geq 0$.

(v) $\langle x, x \rangle = 0$ *if and only if* $x = 0$.

Example 2 *: (Dot product in R^n)*

$$x \cdot y = x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Properties

- $\|x\|_{\mathcal{X}}^2 = \langle x, x \rangle$.
- Schwarz inequality:

$$|\langle x, y \rangle| \leq \|x\|_{\mathcal{X}} \, \|y\|_{\mathcal{X}} \quad \forall x, y \in \mathcal{X}. \tag{2}$$

Example 3 *: (the most important for our purposes)*

$$\langle x, y \rangle = \int_0^\infty x(t) \cdot y(t) \, dt \tag{3}$$

with this inner product, we have that $\mathcal{X} = \mathcal{L}_2$, and moreover

$$\|x\|_{\mathcal{L}_2}^2 = \langle x, x \rangle = \int_0^\infty \|x(t)\|_2^2 \, dt. \tag{4}$$

Notation:

$$\langle x_T, y \rangle = \langle x, y_T \rangle = \langle x_T, y_T \rangle \stackrel{def}{=} \langle x, y \rangle_T. \tag{5}$$

Definition 2 *: (Passivity) A system $H : \mathcal{X}_e \to \mathcal{X}_e$ is said to be passive if*

$$\langle u, Hu \rangle_T \geq \beta \qquad \forall u \in \mathcal{X}_e, \ \forall T \in R^+. \tag{6}$$

Definition 3 *: (Strict Passivity) A system $H : \mathcal{X}_e \to \mathcal{X}_e$ is said to be strictly passive if there exists $\delta > 0$ such that*

$$\langle u, Hu \rangle_T \ \geq \ \delta \|u_T\|_{\mathcal{X}}^2 + \beta \qquad \forall u \in \mathcal{X}_e, \ \forall T \in R^+. \tag{7}$$

$$\beta \text{ is some constant}$$

3 Interconnections of Passivity Systems

Theorem 1 : *Consider systems $H_i : \mathcal{X}e \to \mathcal{X}e, i = 1, \cdots, n$. We have*

(i) If all of the $H_i's, i = 1, \cdots, n$ are passive, then the system $H = H_1 + \cdots + H_n$ is passive.

(ii) If all the systems $H_i, i = 1, \cdots, n$ are passive, and at least one of them is strictly passive, then the system H is strictly passive.

(iii) (feedback systems) If the systems $H_i, i = 1, 2$ are passive then, the mapping from u into y defined by equations (8)–(9) below, is passive.

$$e = u - H_2 y \tag{8}$$

$$y = H_1 e \tag{9}$$

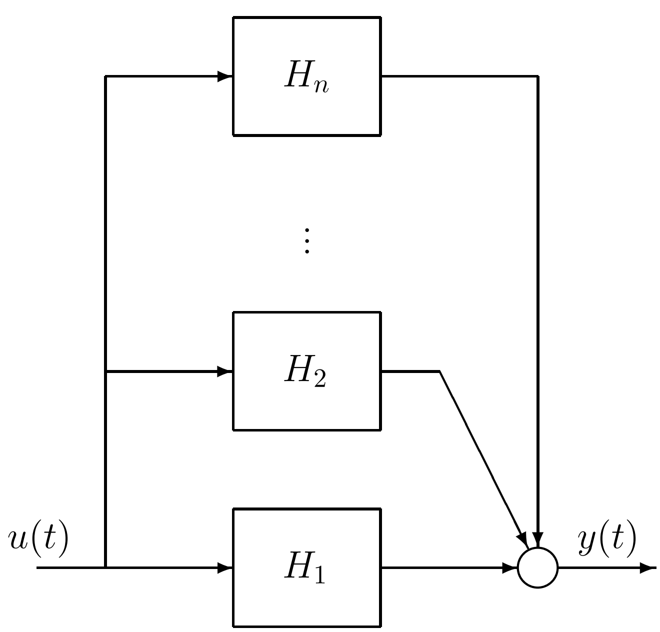

Figure 1: $H = H_1 + H_2 + \cdots + H_n$.

Proof of Theorem 1

Proof of (i): We have

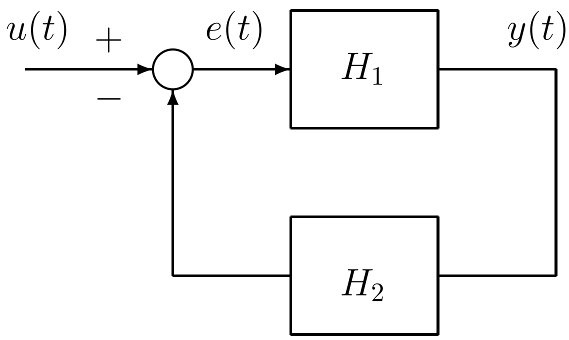

Figure 2: The Feedback System S_1.

$$
\begin{aligned}
\langle x, (H_1 + \cdots + H_n)x\rangle_T &= \langle x, H_1 x + \cdots + H_n x\rangle_T \\
&= \langle x, H_1 x\rangle_T + \cdots + \langle x, H_n x\rangle_T \\
&\geq \beta_1 + \cdots + \beta_n \;\triangleq\; \beta.
\end{aligned}
$$

Thus, $H \triangleq (H_1 + \cdots + H_n)$ is passive.
Proof of (ii) Assume that k out of the n systems H_i are strictly passive, $1 \leq k \leq n$. We can assume that these are the systems $H_1, H_2, \cdots, H_k$. It follows that

$$
\begin{aligned}
\langle x, Hx\rangle_T &= \langle x, H_1 x + \cdots + H_n x\rangle_T \\
&= \langle x, H_1 x\rangle_T + \cdots + \langle x, H_k x\rangle_T + \cdots + \langle x, H_n x\rangle_T \\
&\geq \delta_1\langle x, x\rangle_T + \cdots + \delta_k\langle x, x\rangle_T + \beta_1 + \cdots + \beta_n \\
&= (\delta_1 + \cdots + \delta_k)\|x_T\|_{\mathcal{X}} + (\beta_1 + \cdots + \beta_n)
\end{aligned}
$$

and the result follows.
Proof of (iii): Consider the following inner product:

$$
\begin{aligned}
\langle u, y\rangle_T &= \langle e + H_2 y, y\rangle_T \\
&= \langle e, y\rangle_T + \langle H_2 y, y\rangle_T \\
&= \langle e, H_1 e\rangle_T + \langle y, H_2 y\rangle_T \geq (\beta_1 + \beta_2).
\end{aligned}
$$

□

3.1 Passivity and Small Gain

In the following theorem $\mathcal{X}_e$ is an inner product space, and the gain of a system $H : \mathcal{X}e \to \mathcal{X}e$ is the gain induced by the norm $\|x\|^2 = \langle x, x \rangle$.

Theorem 2 *: Let $H : \mathcal{X}e \to \mathcal{X}e$, and define the function $S : \mathcal{X}e \to \mathcal{X}e$:*

$$S = (H - I)(I + H)^{-1}. \tag{10}$$

We have:

(a) H is passive if and only if the gain of S is at most 1, that is, S is such that

$$\|(Sx)_T\|_{\mathcal{X}} \leq \|x_T\|_{\mathcal{X}} \qquad \forall x \in \mathcal{X}e, \forall T \in \mathcal{X}_e. \tag{11}$$

(b) H is strictly passive and has finite gain if and only if the gain of S is less than 1.

4 Stability of Feedback Interconnections

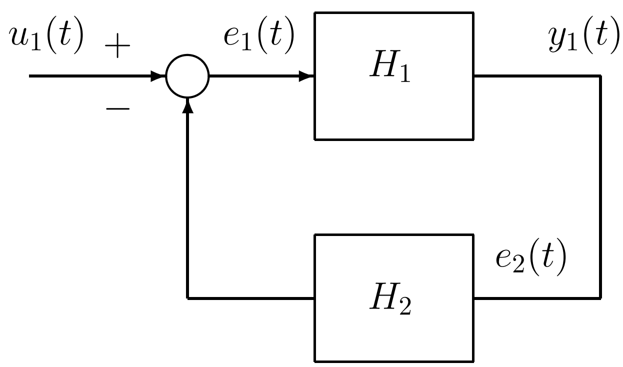

Figure 3: The Feedback System S_1.

Theorem 3 *: Let $H_1, H_2 : \mathcal{X}_e \to \mathcal{X}_e$ and consider the feedback interconnection defined by the following equations:*

$$e_1 = u_1 - H_2 e_2 \tag{12}$$

$$y_1 = H_1 e_1. \tag{13}$$

Under these conditions, if H_1 is passive and H_2 is strictly passive, then $y_1 \in \mathcal{X}$ for every $u_1 \in \mathcal{X}$.

Proof: We have

$$\begin{aligned} \langle u_1, y_1 \rangle_T &= \langle u_1, H_1 e_1 \rangle_T \\ &= \langle e_1 + H_2 e_2, H_1 e_1 \rangle_T \\ &= \langle e_1, H_1 e_1 \rangle_T + \langle H_2 e_2, H_1 e_1 \rangle_T \\ &= \langle e_1, H_1 e_1 \rangle_T + \langle H_2 y_1, y_1 \rangle_T \end{aligned}$$

but

$$\begin{aligned} \langle e_1, H_1 e_1 \rangle_T &\geq 0 \\ \langle H_2 y_1, y_1 \rangle_T &\geq \delta \|y_{1T}\|_{\mathcal{X}}^2 \end{aligned}$$

since H_1 and H_2 are passive and strictly passive, respectively. Thus

$$\langle u_1, y_1 \rangle_T \geq \delta \|y_{1T}\|_{\mathcal{X}}^2$$

By the Schwarz inequality, $|\langle u_1, y_1\rangle_T| \leq \|u_{1T}\|_{\mathcal{X}}\|y_{1T}\|_{\mathcal{X}}$. Hence

$$\begin{aligned} \|u_{1T}\|_{\mathcal{X}}\|y_{1T}\|_{\mathcal{X}} &\geq \delta\|y_{1T}\|_{\mathcal{X}}^2 \\ \Rightarrow \|y_{1T}\|_{\mathcal{X}} &\leq \delta^{-1}\|u_{1T}\|_{\mathcal{X}} \end{aligned} \tag{14}$$

Therefore, if $u_1 \in \mathcal{X}$, we can take limits as T tends to infinity on both sides of inequality (14) to obtain

$$\|y_1\| \geq \delta^{-1}\|u_1\|$$

which shows that if u_1 is in $\mathcal{X}$, then y_1 is also in $\mathcal{X}$. □

Remarks: As stated, this theorem does not guarantee that the error e_1 and the output y_2 are bounded.

Theorem 4 *: If both systems are passive and one of them is (i) strictly passive and (ii) has finite gain, then* $e_1,\ e_2,\ y_1,$ *and* y_2 *are in* $\mathcal{X}$ *whenever* $x \in \mathcal{X}$.

Proof: Very similar.

5 Passivity of Linear Time-Invariant Systems

Theorem 5 *: Consider a linear time-invariant system $H : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ with transfer function $H(s) = n(s)/d(s)$, and assume that the roots of $d(s)$ lie in the left half of the complex plane. We have*

(i) H is passive if and only if $\Re e[\widehat{H}(\jmath\omega)] \geq 0 \quad \forall \omega \in R$.

(ii) H is strictly passive if and only if $\exists \delta > 0$ such that $\Re e[\widehat{H}(\jmath\omega)] \geq \delta \quad \forall \omega \in R$.

Remarks Notice that Theorem 5 applies to systems with no poles on the imaginary axis.

Example 4 *: Consider the system $H : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ defined by its transfer function*

$$\widehat{H}(s) = \frac{\alpha s}{s^2 + \omega_0^2} \quad \alpha > 0, \ \omega_0 \geq 0.$$

Under these conditions, H is passive.

By Theorem 2, H is passive if and only if

$$\|\widehat{S}\|_\infty = \left\| \frac{1-H}{1+H} \right\|_\infty \leq 1$$

but for the given $\widehat{H}(s)$

$$\begin{aligned} \widehat{S}(s) &= \frac{s^2 - \alpha s + \omega_0^2}{s^2 + \alpha s + \omega_0^2} \\ \Rightarrow \quad \widehat{S}(\jmath\omega) &= \frac{(\omega_0^2 - \omega^2) - \jmath\alpha\omega}{(\omega_0^2 - \omega^2) + \jmath\alpha\omega} \end{aligned}$$

which has the form $(a - \jmath b)/(a + \jmath b)$. Thus $|\widehat{S}(\jmath\omega)| = 1 \quad \forall\omega$, which implies that $\|\widehat{S}\|_\infty = 1$, and the theorem is proved. □

Remarks: Systems with a transfer function of this form are oscillatory. In particular, a linear time-invariant model of a flexible structure has the following form:

$$\widehat{H}(s) = \sum_{i=1}^{\infty} \frac{\alpha_i s}{s^2 + \omega_i^2}. \tag{15}$$

Examples of flexible structures include flexible manipulators and space structures. It follows from Theorem 1 that, working in the space $\mathcal{L}_2$, a system with transfer function as in (15) is passive.