Chapter 9: Dissipativity

1 Dissipative Systems

We will consider systems of the form

$$\psi : \begin{cases} \dot{x} = f(x,u), & u \in \mathcal{U}, \quad x \in X \\ y = h(x,u), & y \in \mathcal{Y} \end{cases} \tag{1}$$

and Define a function $w(t) = w(u(t), y(t)) : \mathcal{U} \times \mathcal{Y} \to \ R$, called the *supply rate*, satisfying $\int_{t_0}^{t_1} |w(t)| \ \ dt \ < \ \infty$,

Definition 1 *: ψ is dissipative with respect to $w(t)$ if there exists $\phi : X \to R^+$, called the storage function, such that*

$$\phi(x_1) \le \phi(x_0) + \int_{t_0}^{t_1} w(t) \quad \mathrm{d}t \quad \forall x \in X, \ u \in \mathcal{U} \tag{2}$$

Inequality (2) is called the dissipation inequality.

- $\phi(\cdot)$: the storage function; $\phi(x(t^*))$ represents the "energy" stored by the system ψ at time t^*.
- $\int_{t_0}^{t_1} w(t) \quad dt$: represents the energy externally supplied to the system ψ during the interval $[t_0, t_1]$.

2 Differentiable Storage Functions

Assuming now that ϕ is continuously differentiable we can proceed as follows:

$$\frac{\phi(x_1) - \phi(x_0)}{t_1 - t_0} \leq \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} w(t) \; dt \tag{3}$$

but

$$\lim_{t_1 \to t_0} \frac{\phi(x_1) - \phi(x_0)}{t_1 - t_0} = \frac{d\phi(x)}{dt} = \frac{\partial \phi(x)}{\partial x} f(x, u)$$

and thus (3) is satisfied if and only if

$$\frac{\partial \phi(x)}{\partial x} f(x, u) \leq w(t) = w(u, y) = w(u, h(x, u)) \quad \forall x, u. \tag{4}$$

(4) is called the differential dissipation inequality.

Definition 2 *: (Dissipativity re-stated) ψ is dissipative with respect to $\omega(t) = \omega(u, y)$ if there exist a continuously differentiable function $\phi : X \to R^+$, called the storage function, satisfying:*

(i) There exist α_1 and $\alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x\|) \leq \phi(x) \leq \alpha_2(\|x\|) \qquad \forall x \in \Re^n$$

(ii)

$$\frac{\partial \phi}{\partial x} f(x, u) \leq \omega(u, y) \qquad \forall x \in \Re^n, \; u \in \Re^m, \quad \textit{and} \;\; y = h(x, u).$$

Example 1 *: A system ψ is input-to-state stable if and only if it is dissipative with respect to the supply rate*

$$\omega(t) = -\alpha_3(\|x\|) + \sigma(\|u\|)$$

where α_3 and σ are class $\mathcal{K}_\infty$ functions.

3 QSR Dissipativity

Consider now an important special form of supply rate

Definition 3 : *Given constant matrices Q, S, and R with Q and R symmetric, we define the supply rate $w(t) = w(u, y)$:*

$$w(t) \ = \ y^T Q y + 2 y^T S u + u^T R u \tag{5}$$

Definition 4 : *ψ is QSR-dissipative if there exist $\phi : X \to R^+$ such that*

$$\int_0^T w(t)\, dt = \langle y, Qy\rangle_T + 2\langle y, Su\rangle_T + \langle u, Ru\rangle_T \ \ \geq \phi(x_1) - \phi(x_0). \tag{6}$$

Special Cases:

1- Passive systems: ψ is passive if and only if it is dissipative with $Q = 0, R = 0$, and $S = \frac{1}{2}I$. Notice that:

$$\langle y, u\rangle_T \tag{7}$$

$$(\text{since} \ \ \phi(x) \ > \ 0 \ \ \forall x, \ \ \text{by assumption})$$

Remarks: Notice that is the stored energy at time $t = 0$.

2- Strictly passive systems: ψ is strictly passive if and only if it is dissipative with $Q = 0, R = -\delta$, and $S = \frac{1}{2}I$. Notice that

$$\langle y, u\rangle_T + \langle u, -\delta u\rangle_T \ \geq \ \phi(x_1) - \phi(x_0) \ \ \geq \ -\phi(x_0) \ \triangleq \ \ \beta$$

or

$$\langle u, y\rangle_T \ \geq \ \delta\langle u, u\rangle_T \ + \ \beta \ = \ \delta\|u\|_T^2 \ + \ \beta.$$

3- Finite-gain-stable: ψ is finite-gain-stable if and only if it is dissipative with $Q = -\frac{1}{2}I, R = \frac{\gamma^2}{2}I$, and $S = 0$:

$$-\frac{1}{2}\langle y, y\rangle_T + \frac{\gamma^2}{2}\langle u, u\rangle_T$$

$$\Rightarrow \ \ \|y_T\|_{\mathcal{L}_2}$$

and defining $\beta = \sqrt{2\phi(x_0)}$, we have

$$\|y_T\|_{\mathcal{L}_2} \ \leq \ \gamma \ \|u_T\|_{\mathcal{L}_2} \ + \ \beta.$$

4- **Strictly output-passive systems**: ψ is strictly output passive if it is dissipative with $Q = -\epsilon I, R = 0$, and $S = \frac{1}{2}I$. Notice that:

$$-\epsilon\langle y, y\rangle_T + \langle y, u\rangle_T \ \geq \ \phi(x_1) - \phi(x_0) \ \ \geq \ -\phi(x_0)$$

or

$$\int_0^T u^T y \, dt \ = \ \langle u, y\rangle_T \ \geq \ \epsilon\langle y, y\rangle_T \ + \ \beta.$$

5- Very strictly-passive Systems: ψ is very strictly passive if it is dissipative with $Q = -\epsilon I, R = -\delta I$, and $S = \frac{1}{2}I$. We have

$$-\epsilon\langle y, y\rangle_T - \delta\langle u, u\rangle_T + \langle y, u\rangle_T \geq \ -\phi(x_0) \ \triangleq \ \beta$$

or

$$\int_0^T u^T y \, dt \ = \ \langle u, y\rangle_T \ \geq \ \delta\langle u, u\rangle_T + \epsilon\langle y, y\rangle_T \ + \ \beta.$$

Lemma 1 *: If ψ is strictly output passive, then it has a finite $\mathcal{L}_2$ gain.*

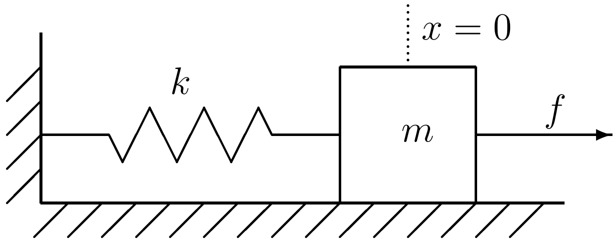

Figure 1: Mass-spring system.

4 Examples

4.1 Mass–Spring System with Friction

Consider the mass–spring system moving on a horizontal surface.

$$m\ddot{x} + \beta\dot{x} + kx = f$$

Defining $x_1 = x,\ \dot{x}_1 = x_2$ we obtain:

$$\psi : \begin{cases} \dot{x}_1 &=& x_2 \\ \dot{x}_2 &=& -\frac{k}{m}x_1 - \frac{\beta}{m}x_2 + \frac{f}{m} \\ y &=& x_2 \end{cases}$$

Defining $\phi = E$, the total energy in the system at time t, we obtain:

$$\phi \triangleq E = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2.$$

$$\dot{\phi} = \frac{\partial\phi}{\partial x}\dot{x} = -\beta y^2 + yf.$$

Thus,

$$\int_0^t \dot{\phi}\, dt \ = \ E(t) \ \geq 0$$

thus ψ is dissipative with respect to the supply rate

$$\omega(t) = yf - \beta y^2.$$

This supply rate corresponds to $Q = -\beta$, $S = \frac{1}{2}$, and $R = 0$. This means that the mass-spring system is strictly output-passive.

4.2 Mass–Spring System without Friction

Consider again the mass–spring system with $\beta = 0$.

$$\psi : \begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 + \frac{f}{m} \\ y &= x_2 \end{cases}$$

Proceeding as in the previous example, we define

$$\phi \triangleq E = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2.$$

Differentiating ϕ along the trajectories of ψ, we obtain:

$$\dot{\phi} = x_2 f = yf$$

since once again,

$$\int_0^t \dot{\phi}\, dt \;=\; E(t) \;\geq 0.$$

We conclude that the mass–spring system with output $\dot{x} = x_2$ is dissipative with respect to

$$\omega(t) = yf$$

This implies that the mass–spring system is passive.

5 Stability of Dissipative Systems

Consider a dissipative system ψ with storage function ϕ. Assume that x_e is an equilibrium point for the unforced system.

Theorem 2 *: Let ψ be dissipative with respect to the storage function $\phi : X \to R^+$ and assume that:*

(i) x_e is a strictly local minimum for ϕ:

$$\phi(x_e) < \phi(x) \quad \forall x \text{ in a neigborhood of } x_e$$

(ii) The supply rate $w = w(u, y)$ is such that

$$w(0, y) \leq 0 \quad \forall y.$$

Under these conditions x_e is a stable equilibrium point for the unforced systems $\dot{x} = f(x, 0)$.

Proof: Define the function $V(x) \triangleq \phi(x) - \phi(x_e)$. This function is continuously differentiable, and by condition (i) is positive definite $\forall x$ in a neighborhood of x_e. Also, the time derivative of V along the trajectories of ψ is given by

$$\dot{V}(x) = \frac{\partial \phi(x)}{\partial x} f(x, u)$$

thus, by (4) and condition (ii) we have that $\dot{V}(x) \leq 0$ and stability follows by the Lyapunov stability theorem. □

Corollary 3 *: If in addition no solution of $\dot{x} = f(x)$ other than $x(t) = x_e$ satisfies $w(0, y) = w(0, h(x, 0) = 0$, then x_e is asymptotically stable.*